Final Exam — MAT 103 — Fall 2007

The first eight questions are True/False questions. Circle your answers and provide brief justifications. Each question is worth 5 points — 3 pts for the correct answer and 2 pts for the justification.

(All of the questions on subsequent pages are worth 10 points each.)

1. True or False: A golden rectangle is more than twice as wide as it is high.

2. True or False: There is a triangle with side lengths 1, 2 and 4.

3. True or False: Any two line segments contain the same number of points.

4. True or False: There are exactly two infinite cardinal numbers.

5. True or False: 27 is a prime number.

6. True or False: 91 is a prime number.

7. True or False: It is impossible to use 3 cent and 12 cent stamps to create exactly 40 cents worth of postage.

8. True or False: An octahedron is so-named because its faces are octagons.

9. Find the power set of $\{A, 1\}$.

10. Consider the following two sets:

    $\mathcal{S} = \{1, 4, 9, 16, 25, 36, \ldots\}$ — (the perfect squares),

    and

    $\mathcal{O} = \{1, 3, 5, 7, 9, 11, \ldots\}$ — (the odd numbers).

    Show that these two sets have the same cardinality by finding an explicit one-to-one correspondence between them.

11. Draw a picture that indicates a one-to-one correspondence showing that the set of points on a circle of radius 1 having a single point removed, has the same cardinality as the set of points on the entire real line.

12. Fill in the blanks in the proof of the following theorem:

Theorem 1. *The set of prime numbers is infinite.*

Proof. Suppose not. Then there is a ______________ list which contains all of the primes. We can list the primes in ascending order.

$$p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7,\ \ldots,\ p_n$$

Consider the integer formed by taking the product of all of these primes and then adding 1.

$$N = (p_1 p_2 p_3 p_4 \ldots p_n) + 1$$

This number is ______________ than any of the primes in our list, in particular it cannot be prime itself and must therefore be divisible by at least one of the primes (say p_j). On the other hand, N is one greater than a multiple of p_j, so p_j cannot divide N. This is a contradiction since we have shown that N is both ______________ and not divisible by p_j. Hence the supposition is false and the theorem is true. □

13. A Pythagorean triple is a set of three whole numbers that satisfy the Pythagorean theorem. For example, in class we have seen that $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples.

    Which of the following are Pythagorean triples?

    (a) $(8, 6, 9)$

    (b) $(10, 24, 25)$

    (c) $(1, 1, 2)$

    (d) $(8, 15, 17)$

    (e) $(7, 24, 25)$

14. Illustrate the construction of a golden rectangle.

15. How many faces, edges and vertices are there on an icosahedron?

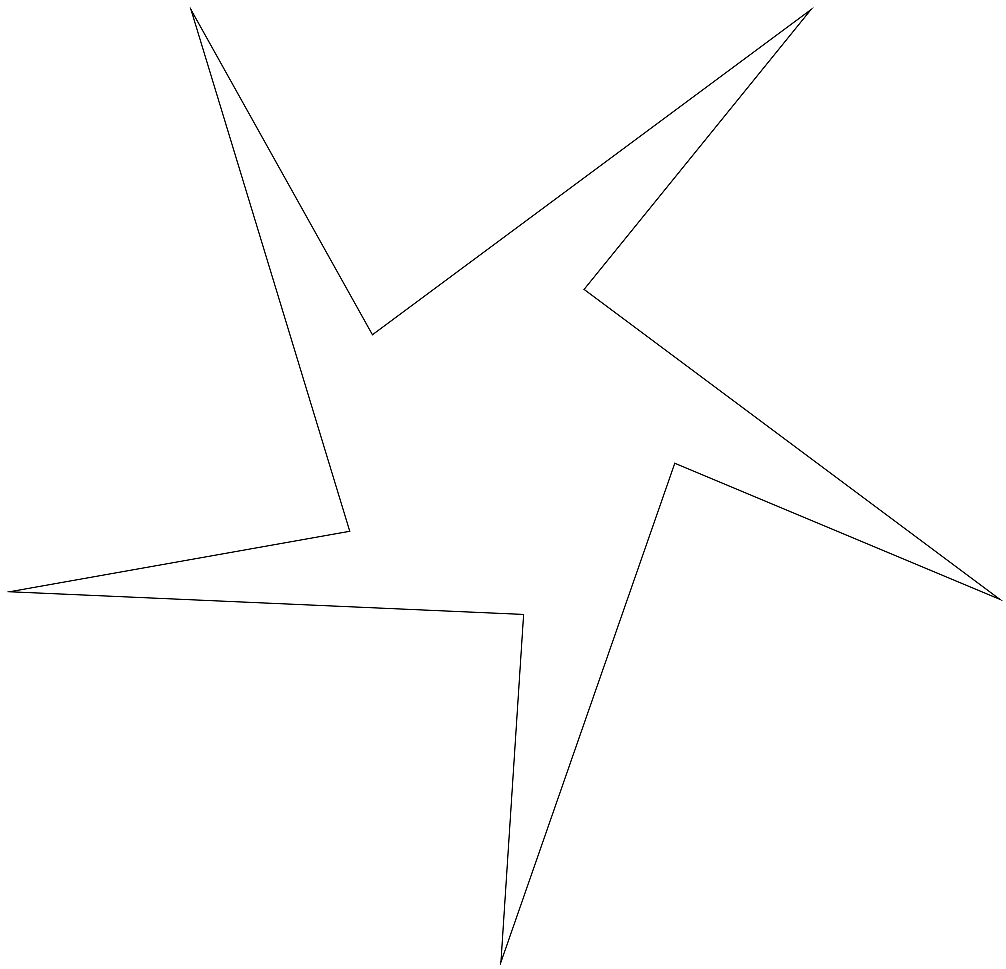
16. What is the straight line distance between the points $(2, 5)$ and $(5, 1)$ in the Euclidean plane?

17. For each number n from 1 to 11 find the value of $2^n \pmod{11}$.

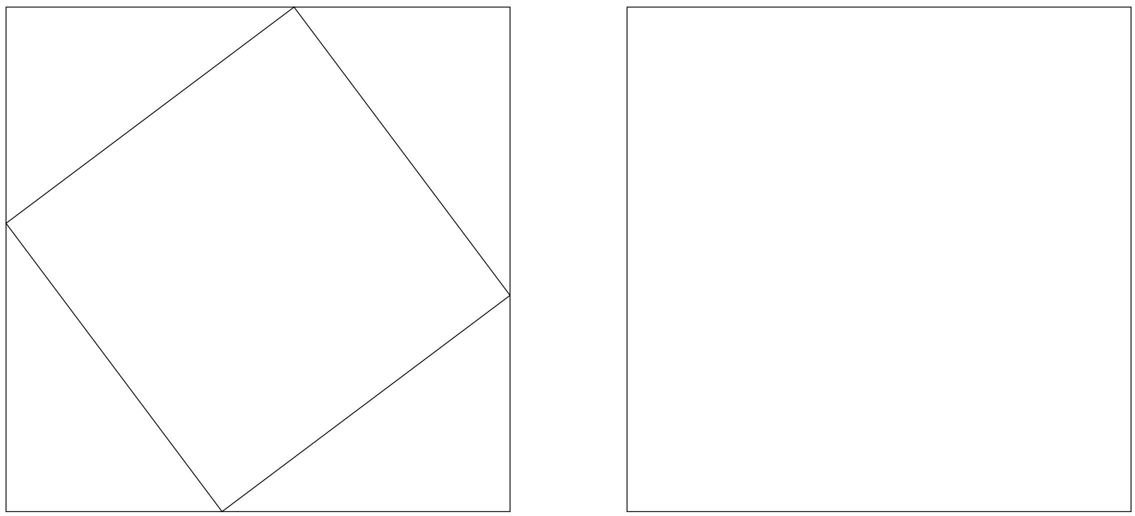
18. The Lucas numbers satisfy the same recursive formula as the Fibonacci numbers but have different initial values. Specifically, $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$.

Find the first 10 Lucas numbers.

19. Use Fisk's algorithm to find positions for security cameras in the following art gallery.



20. The figure below is incomplete. Finish the picture by adding lines to the right-hand square so that the proof of the Pythagorean theorem is illustrated.



21. How can we use unmarked containers that hold 6 ounces and 10 ounces respectively to measure out precisely 8 ounces of water from a running tap.

22. An RSA public-key cryptosystem is to be implemented using the prime numbers 43 and 47. The encoding exponent will be chosen so that it has no factors in common with 42 and 46. We will use 5 as the encoding exponent, so the public key consists of $(43 \cdot 47, 5)$ or $(2021, 5)$.

   (a) Find an encoding exponent other than 5 which would be usable.

   (b) Verify that 773 is the correct decoding exponent to use (when the encoding exponent is 5).

   (c) How would we encode the message "1001" using this public key?

   (d) If the encrypted message "2001" was received, how would we decode it?

23. Recall that in Russell's Paradox we showed that the notion of a set that contains all sets leads to a contradiction. This contradiction is created by considering the set of amenable sets — the set of all sets that do not contain themselves (as elements).

   We then ask the "inconvenient question": Is the set S itself an amenable set? In other words is $S \in S$?

   What precisely is the contradiction we then obtain?

24. There are 6,010 full-time undergraduate students at SCSU. True or False: there must be two students whose weights are identical to the nearest ounce. Justify your answer. (Recall that there are 16 ounces in one pound. Let us say for the sake of this problem that no student's weight exceeds 350 pounds.)