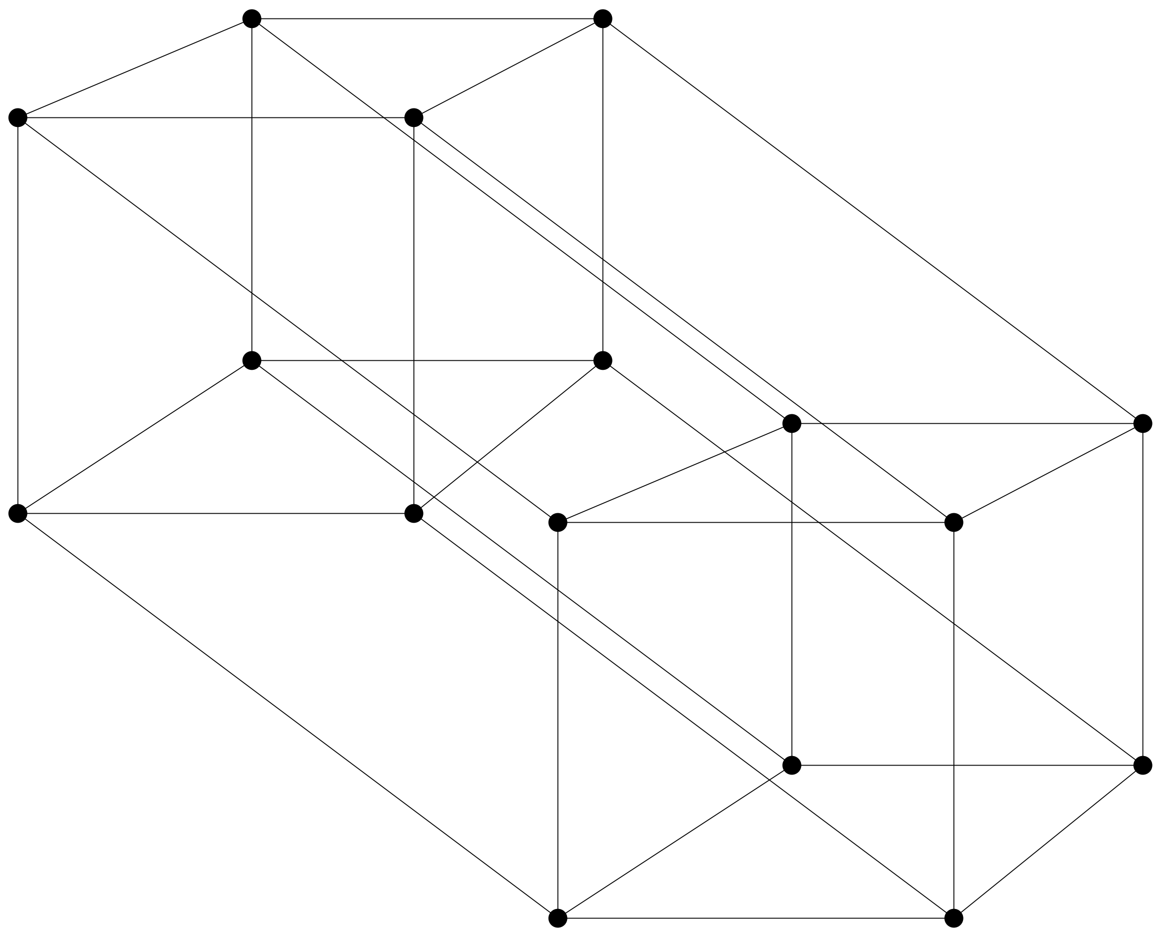
Final Sample — MAT 378 — Spring 2008

(1) Find a Hamiltonian cycle on the edge-graph of the hypercube.



(2) How many distinct solutions are there to the equation $\sum_{i=1}^{50} x_i = 100$ if $0 < x_i$ is a strictly positive integer for all i? Express your answer as a binomial coefficient.

(3) A sequence is defined recursively by

$$a_n = 5 \cdot a_{n-1} - 6 \cdot a_{n-2}$$

together with the initial data $a_0 = 0$ and $a_1 = 1$.

Find a formula which gives a_n in terms of n.

(4) The American Standard Code for Information Interchange (ASCII) defines meanings for 128 binary 7-tuples. Usually a so-called parity check bit is added so that ASCII symbols correspond to *bytes*, or strings of 8 bits. For a cryptographic application we might be asked to consider the set of all 5 letter ASCII strings. (Where we are using the term "letter" to refer to one of the ASCII symbols.)

(a) How many 5 letter ASCII words are there in total?

(b) How many 5 letter ASCII words have no repeated symbols?

(c) How many 5 letter ASCII words have the symbols occuring in increasing numerical order? (Each ASCII symbol may be thought of as a number from 0 to 127.)

(d) How many 5 letter ASCII words have the symbols occuring in non-decreasing numerical order?

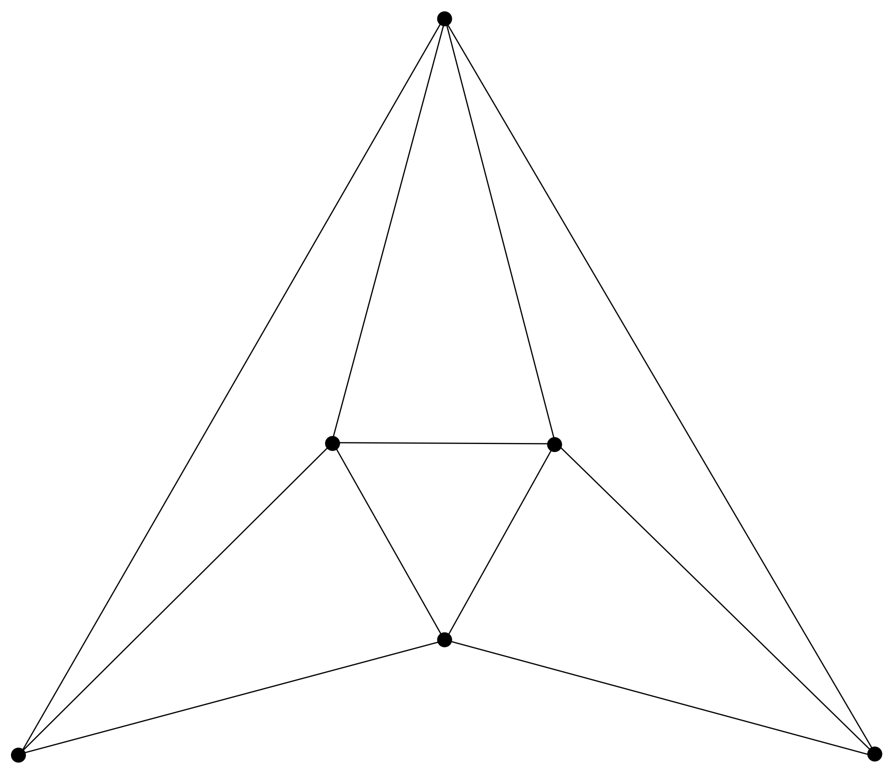
(5) Give a combinatorial proof of

$$r \cdot \binom{n}{k} \cdot \binom{k}{r} \quad = \quad n \cdot \binom{n-1}{r-1} \cdot \binom{n-r}{k-r}.$$

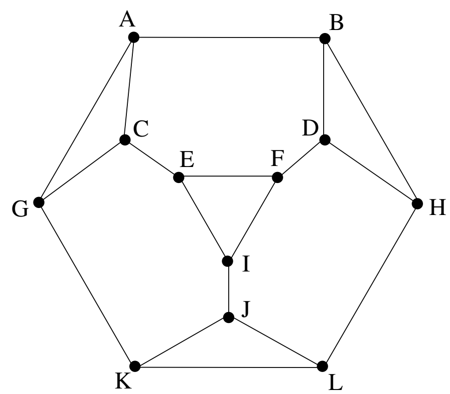
(6) Sheila is sorting tennis balls by color. She has a container that holds 50 balls into which she is putting green balls, a container that holds 20 balls into which she is putting orange balls and a container that holds 10 balls into which she is putting white balls. What is the minimum number of balls (of any colors) which would *guarantee* that one of her containers is full?

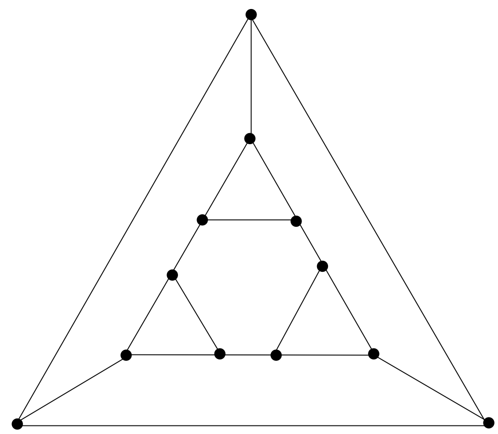
(7) If 10 people put their car keys in a basket and then take turns picking out keys at random, what is the probability that *no one* ends up with their own keys?

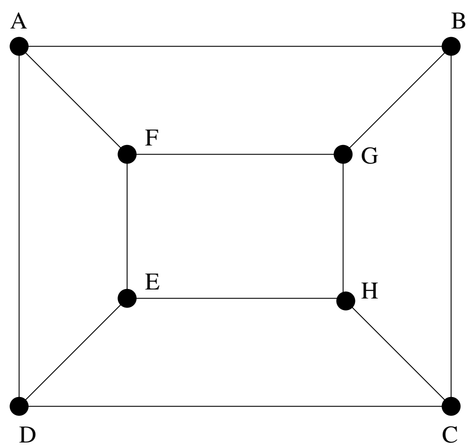
(8) Does the following graph have an Eulerian circuit or path? If so, indicate it, if not, explain why not.

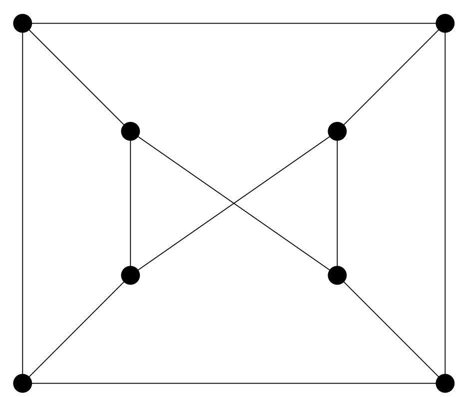


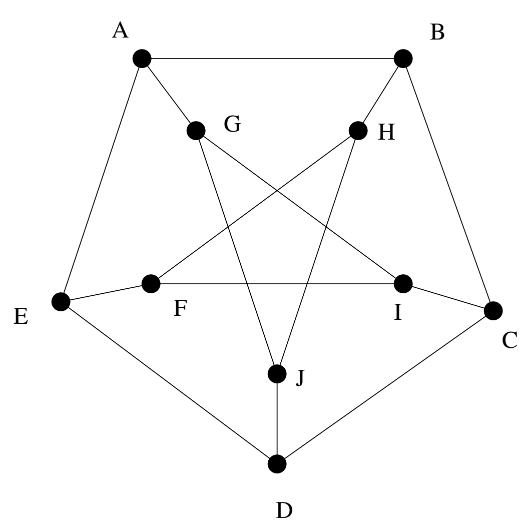
(9) Consider the following pairs of graphs. If there is an isomorphism, indicate it by labelling the nodes of the right-hand graph — if not, name a graph invariant that the two graphs do not have in common.

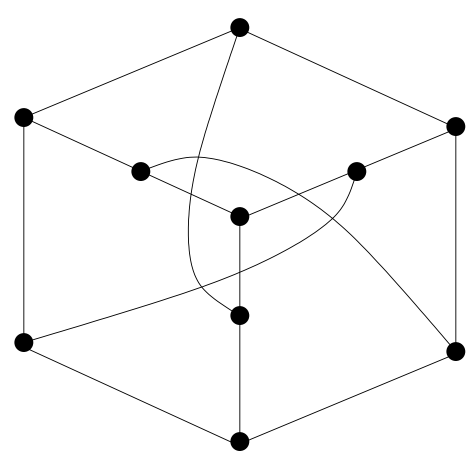








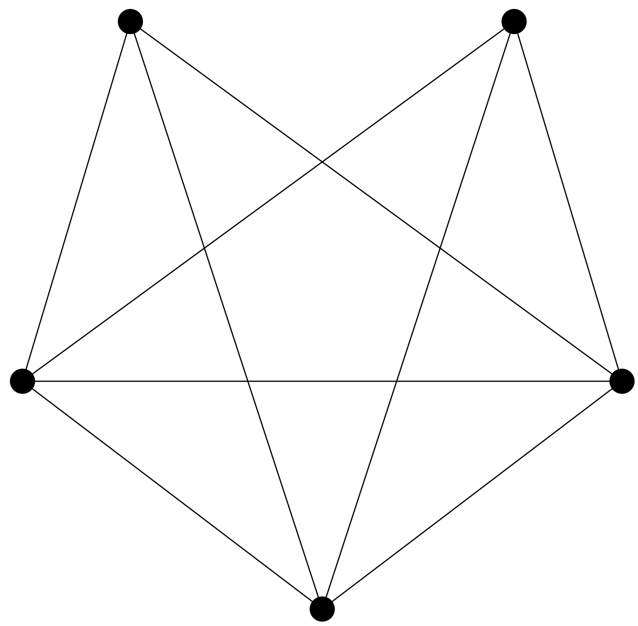




(10) Determine a cyclic Gray code of order 4 (an ordering of the 16 binary 4-tuples where successive entries are at Hamming distance 1, as are the first and last) that is *not* the reflected Gray code order.

(11) List all 6 of the permutations of $\{1, 2, 3\}$ and give their inversion sequences $[a_1, a_2, a_3]$.

(12) Consider the graph formed by deleting one edge from a K_5. What is its chromatic number?



(13) Give an algebraic proof of Pascal's formula for binomial coefficients —

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$