Pierre de Fermat (1601–65)

Pierre Fermat came from a prosperous bourgeois family. His father, Dominique, was the proprietor of a successful leather business; his mother, Claire de Long, was from a family of jurists, belonging to the "noblesse de robe." Pierre loved classical studies and proved remarkably adept in them. A master of Greek and Latin, he studied classical literature, dabbled in philosophy, and composed Latin poetry. He also studied two modern languages, Italian and Spanish. He probably attended the University of Bordeaux before receiving a degree in civil laws from the University of Orleans in 1631. In the same year, he married his cousin Louise de Long. They had five children.

Because mathematics did not become a profession with employment opportunities and scholarly organizations for another two centuries, Fermat followed mathematics only as an avocation and made his living as a jurist. In 1631, he purchased the offices of counselor and master of requests at the Toulouse *parlement* (high court). At that time judicial offices were sold in France. Principally because of a high rate of mortality among his colleagues, Fermat rose rapidly in the legal profession. In 1638, he became *conseiller aux enquêtes* in Toulouse, and four years later he was named to the highest councils of the Toulouse *parlement*—the Criminal Court and then the Grand Chamber (acting as the latter's chief spokesman in 1648). However, certain critics viewed his legal work as occasionally less than satisfactory. In 1664, the Languedoc *intendant* (local royal administrative official) described him in deprecating terms to Jean-Baptiste Colbert, the finance minister to Louis XIV. Probably by reason of seniority, Fermat, a staunch Catholic, presided over the Chambre de l'Edit, which had jurisdiction in suits between Huguenots and Catholics.

Throughout his life Fermat proved an extraordinary mathematical genius, especially in the origins of infinitesimal calculus and the theory of numbers, where he was without a peer for more than a century. He presented his important conceptions and discoveries in correspondence with his friends, primarily Bernard Frénicle de Bessy, Pierre Carcavi, Pierre Gassendi, Christian Huygens, Marin Mersenne, and Gilles Roberval, rather than in publications. He enjoyed good relations with several other savants in Paris, among whom he circulated handwritten manuscripts, but copies of these are less faithful to Fermat's ideas than his correspondence. In 1670 his son, Samuel, edited and enriched his father's notes on Diophantus, which contain many of his discoveries in the theory of numbers. In 1679, Samuel edited *Varia Opera,* a collection of his father's writings.

Fermat independently discovered coordinate (or analytic) geometry and contributed to the immediate origins of infinitesimal calculus. A disciple of François Viète since his days in Bordeaux, he early pursued symbolic algebra and defended his master's algebraic notation, even when Descartes rendered it obsolete. Fermat was a cultural conservative whose study of ancient sources—including the extant writings of Archimedes, Apollonius, Euclid, Pappus, and, above all, the *Arith-*

metica of Diophantus, which was published in a Latin translation in 1621—also influenced his research. By 1636, he had devised a system of analytic geometry that he recorded in his brief treatise, *Ad locos planos et solidos isagoge* ("Introduction to Plane and Solid Loci," publ. posth. 1679). His presentation is more direct and systematic than Descartes's version in *La géométrie* and, except for its notation, much closer to the modern version. Fermat differs from Descartes by beginning with plotting curves or loci from equations instead of with geometric constructions. In general, neither man plotted curves in a rectangular coordinate system but generated them from a single axis with a moving ordinate. Viewing Fermat's methods as rivals to his own, Descartes, who was not tolerant of rivals, asserted in 1638 that they were paralogistic in their reasoning and had only limited application. These assertions initiated an acrid priority dispute in analytic geometry. In other work, Fermat's method of maxima and minima evolved by 1637 into an algorithm (rule of mathematical procedure) that is equivalent to differentiation. To obtain maxima and minima points of curves; he projected changes in the x-axis by $x + E$, and he had E go to zero at these inflexion points. With this procedure he was able to find the equations of tangents. As a result of this analysis of curves and his building upon the work of Archimedes on summation processes for quadrature, including upper and lower limits, Fermat came very close to discovering infinitesimal calculus.

Today Fermat is probably best known for his pioneering work in his favorite field, number theory, where he concentrated on primeness and divisibility. Based on his finding that numbers of the form $2^x + 1$, where $x = 2^n$ and n is an integer are prime for $n = 1$ through 4, he conjectured that all such numbers are prime. (In the eighteenth century Leonhard Euler showed this to be false for $n = 5$, or $2^{32} - 1$. Numbers of this form are not prime for n from 5 through 16.) In a 1640 letter, Fermat also essentially stated but did not prove his lesser or little theorem: For p prime and a and p relatively prime, $(a^{p-1} - 1)$ is divisible by p. In modern notation $a^{p-1} \equiv 1 \bmod p$. A variation of this theorem without the divisibility restriction on a and p is: For any number a and prime p, p divides $a^p - a$. Leibniz proved this theorem in a manuscript before 1683, and Euler offered his first proof in the St. Petersburg *Commentarii* in 1736. Fermat's last or great theorem asserts that there are no positive integers x, y, and z, such that $x^n + y^n = z^n$ when $n > 2$. Referring to it in 1637, Fermat wrote on page 241 of his copy of Claude-Gaspard Bachet's book, *Diophanti,* "For this, I have discovered a truly remarkable proof but this margin is too narrow to contain it." Subsequent historical research suggests that this statement in *Diophanti* is not correct. The last theorem defied solution for more than 350 years. After the Shimura-Taniyama-Weil conjecture concerning semistable elliptic curves in 1971 and Berkeley mathematician Kenneth Ribet's demonstration that Fermat's last theorem is a corollary of this result in the theory of elliptic curves, British mathematician Andrew Wiles of Princeton in 1993 essentially proved the Shimura-Taniyama-Weil conjecture and its corollary of Fermat's last theorem.

Through correspondence with Blaise Pascal, Fermat also helped to create the foundation for probability theory. The two cofounders proceeded differently. Fermat utilized a superior approach that relied on direct computation rather than general mathematical formulas. Nevertheless, Christian Huygens soon replaced his probabilistic methods with more sophisticated ones in *De ludo aleae* (1657). Fermat vainly tried to interest Pascal in number theory. In the 1640s, some commentators associated Fermat's study of different forms of numbers with the vogue in Paris for magic and astrology. In the 1660s Fermat planned to meet Pascal, but the meeting had to be canceled when both were too ill to travel.

Fermat's interest in mathematics extended to diophantine equations and their application to optics. In the late 1650s and 1660s, he investigated the laws of refraction. His belief that light does not travel faster in denser media led to a polemic with the Cartesians. Fermat postulated *his beautiful principle of least time* in optics: A ray of light traveling between two points always takes the path requiring least time. With this principle he used early roots of the calculus of variations, as Euler and Lagrange illustrated in the eighteenth century.

69

From “On the Transformation and Simplification of the Equations of Loci” (ca. 1640)[1*]

(Integration)

– PIERRE DE FERMAT

Applications of the Geometric Progression to the Quadrature[2] of Parabolas and Infinite Hyperbolas

Archimedes did not employ geometric progressions except for the quadrature of the parabola; in comparing various quantities he restricted himself to arithmetic progressions. Was this because he found that the geometric progression was less suitable for the quadrature? Was it because the particular device that he used to square the parabola by this progression can only with difficulty be applied to other cases? Whatever the reason may be, I have recognized and proved that this progression is very useful for quadratures, and I am willing to present to modern mathematicians my invention which permits us to square, by a method absolutely similar, parabolas as well as hyperbolas.

The entire method is based on a well-known property of the geometric progression, namely the following theorem:

> *Given a geometric progression the terms of which decrease indefinitely, the difference between two consecutive terms of this progression is to the smaller of them as the greater one is to the sum of all following terms.*[3]

This established, let us discuss first the quadrature of hyperbolas:

I define hyperbolas as curves going to infinity, which, like *DSEF* [Fig. 69.1], have the following property. Let *RA* and *AC* be asymptotes which may be extended indefinitely; let us draw parallel to the asymptotes any lines *EG, HI, NO, MP, RS,* etc. We shall then always have the same ratio between a given power of *AH* and the same power of *AG* on one side, and a power of *EG* (the same as or different from the preceding) and the same

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* SOURCE: This translation made from the *Oeuvres de Fermat*, edited by P. Tannery and C. Henry, I (1891), 255–59 and III (1896), 216–19, appears in D. J. Struik, ed., *A Source Book in Mathematics, 1200–1800* (1969), 219–22. It is reprinted with permission of Harvard University Press. Copyright © 1969 by the President and Fellows of Harvard College. Footnotes renumbered. The biography of Pierre de Fermat precedes selection 64.

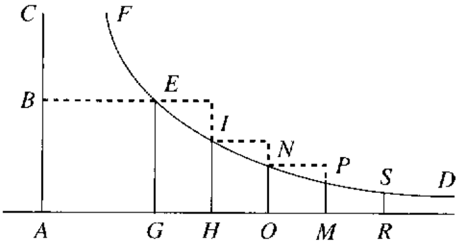

Figure 69.1

power of HI on the other. I mean by powers not only squares, cubes, fourth powers, etc., the exponents of which are 2, 3, 4, etc., but also simple roots the exponent of which is unity.[4]

I say that all these infinite hyperbolas except the one of Apollonius,[5] *or the first, may be squared by the method of geometric progression according to a uniform and general procedure.*

Let us consider, for example, the hyperbolas the property of which is defined by the relations $AH^2/AG^2 = EG/HI$ and $AO^2/AH^2 = HI/NO$, etc. I say that the indefinite area which has for base EG and which is bounded on the one side by the curve ES and on the other side by the infinite asymptote GOR is equal to a certain rectilinear area.

Let us consider the terms of an indefinitely decreasing geometric progression; let AG be the first term, AH the second, AO the third, etc. Let us suppose that those terms are close enough to each other that following the method of Archimedes we could adequate [*adégaler*] according to Diophantus,[6] that is, equate approximately the rectilinear parallelogram $GE \times GH$ and the general quadrilateral $GHIE$; in addition we shall suppose that the first intervals GH, HO, OM, etc. of the consecutive terms are sufficiently equal that we can easily employ Archimedes' method of exhaustion by circumscribed and inscribed polygons. It is enough to make this remark once and we do not need to repeat it and insist constantly upon a device well known to mathematicians.

Now, since $AG/AH = AH/AO = AO/AM$, we have also $AG/AH = GH/HO = HO/OM$, for the intervals. But for the parallelograms,

$$\frac{EG \times GH}{HI \times HO} = \frac{HI \times HO}{ON \times OM}.$$

Indeed, the ratio $EG \times GH/HI \times HO$ of the parallelograms consists of the ratios EG/HI and GH/HO; but, as indicated, $GH/HO = AG/AH$; therefore, the ratio $EG \times GH/HI \times HO$ can be decomposed into the ratios EG/HI and AG/AH. On the other hand, by construction, $EG/HI = AH^2/AG^2$ or AO/AG, because of the proportionality of the terms; therefore, the ratio $GE \times GH/HI \times HO$ is decomposed into the ratios AO/AG and AG/GH; now AO/AH is decomposed into the same ratios; we find consequently for the ratio of the parallelograms: $EG \times GH/HI \times HO = AO/AH = AH/AG$.

Similarly we prove that $HI \times HO/NO \times MO = AO/AH$.

But the lines AO, AH, AG, which form the ratios of the parallelograms, define by their construction a geometric progression; hence the infinitely many parallelograms $EG \times GH$, $HI \times HO$, $NO \times OM$, etc., will form a geometric progression, the ratio of which will be AH/AG. Consequently, according to the basic theorem of our method, GH, the difference of two consecutive terms, will be to the smaller term AG as the first term of the progression, namely, the parallelogram $GE \times GH$, to the sum of all the other parallelograms in infinite number. According to the adequation of Archimedes, this sum is the infinite figure bounded by HI, the asymptote HR, and the infinitely extended curve IND.

Now if we multiply the two terms by EG we obtain $GH/AG = EG \times GH/EG \times AG$; here $EG \times GH$ is to the infinite area the base of which is HI as $EG \times GH$ is to $EG \times AG$. Therefore, the parallelogram $EG \times AG$, which is a given rectilinear area, is adequated to the said figure; if we add on both sides the parallelogram $EG \times GH$, which, because of infinite subdivisions, will vanish and will be reduced to nothing, we reach a conclusion that would be easy to confirm by a more lengthy proof carried out in the manner of Archimedes, namely, that for this kind of hyperbola the parallelogram AE is equivalent to the area bounded by the base *EG*, the asymptote *GR*, and the curve *ED* infinitely extended.

It is not difficult to extend this idea to all the hyperbolas defined above except the one that has been indicated.

Commentary by D. J. Struik

Fermat then extends his method to parabolas. His reasoning can be translated as follows.

Divide the interval $0 \leq x < a$ into parts by the points $x_1 = a$, $x_2 = ar$, $x_3 = ar^2, \ldots, r < 1$, which are separated by the intervals $l_1 = a(1 - r)$, $l_2 = ar(1 - r)$, $l_3 = ar^2(e - r), \ldots$. If $y = x^n$ ($n = p/q$, $p, q \lessgtr 0$) is the equation of the "hyperbola" or "parabola," then the values of y corresponding to $x_1, x_2, x_3, \ldots$ are $y_1 = a^n$, $y_2 = a^n r^n$, $y_3 = a^n r^{2n}, \ldots$. Then the sum S of the rectangles $l_1x_1 + l_2x_2 + l_3x_3 + \cdots$ is

$$
\begin{aligned}
S &= a(1 - r)a^n + ar(1 - r)a^n r^n + ar^2(1 - r)a^n r^{2n} \\
&\quad + \cdots \\
&= (1 - r)a^{n+1}(1 + r^{n+1} + r^{2n+2} + \cdots) \\
&= \frac{1 - r}{1 - r^{n+1}}\, a^{n-1}.
\end{aligned}
$$

When $r = s^q$ ($s < 1$) and $n \neq -1$, then

$$
\int_o^a x^n\, dx = a^{n+1} \lim \frac{1 - r}{1 - r^{n-1}} = a^{n-1} \lim \frac{1 - s^q}{1 - s^{p+q}}
$$

$$
= \frac{qa^{n-1}}{p + q} = \frac{a^{n+1}}{n + 1}.
$$

As we see, this procedure holds for n positive and negative, but it fails for $n = -1$.

This method approaches our modern method of limits; it uses the concept of the limit of an infinite geometric series.

Struik's Notes

1. (*Editor's Note*). This paper generalizes Bonaventura Cavalieri's integral, which in modern symbols is

$$\int_0^a x^n dx = \frac{a^{n-1}}{n + 1},$$

where n is any positive whole number, to n fractional or negative.

2. Fermat uses the Greek term *tetragonizein* for "to perform a quadrature," a practice not uncommon in the seventeenth century.
3. This is Fermat's way of expressing that the sum of a convergent series $a + ar + ar^2 + \cdots + ar^n + \cdots = a/(1 - r)$.
4. This may mean "exponents that are unit fractions."
5. The hyperbola of Apollonius is the ordinary hyperbola, of which, if its equation is $xy = a^2$, the integral $\int_0^\infty y\, dx$ diverges.
6. The term *adequatio* is a Latin translation of the Greek term *parisótēs*, by which Diophantus denoted an approximation to a certain number as closely as possible. See T. L. Heath, *Manual of Greek Mathematics* (Clarendon Press, Oxford, 1931), 493. Fermat uses the term to denote what we call a limiting process.