we must determine which are the prime numbers that factor their first power minus 1 in such a way that the exponent of the said power be an odd number—which I think very difficult [*fort malaisé*].

Fermat then continues with other striking properties of powers, also of numbers of the form $2^n + 1$, which, he believed, are all prime if n is a power of 2.[3]

7 FERMAT. THE "PELL" EQUATION

In a letter of February 1657 (*Oeuvres*, II, 333–335; III, 312–313) Fermat challenged all mathematicians (thinking probably in the first place of John Wallis in England) to find an infinity of integer solutions of the equation $x^2 - Ay^2 = 1$, where A is any nonsquare integer. He may have been led to this by his study of Diophantus, who set the problem of finding, for example, a number x such that both $10x + 9$ and $5x + 4$ are squares. If these squares are called u^2 and v^2 respectively, then $u^2 - 2v^2 = 1$, and a solution is $x = 28$. The problem was taken up by De Billy (see below) and later by Euler, who in his "De solutione problematum Diophanteorum per numeros integros," *Commentarii Academiae Scientiarum Petropolitanae 6* (1732/33, publ. 1738), 175–188, *Opera omnia*, ser. I, vol. 2, 6–17, referred to the problem as that of Pell and Fermat. John Pell (1611–1685), an English mathematician, had little to do with the problem, but the problem of Fermat has since been known as that of the Pell equation. It had already been studied by Indian mathematicians, and even in the *Cattle Problem*, attributed to Archimedes, which leads to a "Pell" equation with $A = 4729494 = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353$; see T. L. Heath, *A manual of Greek mathematics* (Clarendon Press, Oxford, 1931), 337.

Fermat, after observing that "Arithmetic has a domain of its own, the theory of integral numbers," defines his problem as follows:

Given any number not a square, then there are an infinite number of squares which, when multiplied by the given number, make a square when unity is added.

Example.—Given 3, a nonsquare number; this number multiplied by the square number 1, and 1 being added, produces 4, which is a square.

Moreover, the same 3 multiplied by the square 16, with 1 added makes 49, which is a square.

And instead of 1 and 16, an infinite number of squares may be found showing the same property; I demand, however, a general rule, any number being given which is not a square.

It is sought, for example, to find a square which when multiplied into 149, 109, 433, etc., becomes a square when unity is added.

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[3] See note 2.

In the same month (February 1657) Fermat, in a letter to Frénicle, suggests the same problem, and expressly states the condition, implied in the foregoing, that the solution be in integers:

> Every nonsquare is of such a nature that one can find an infinite number of squares by which if you multiply the number given and if you add unity to the product, it becomes a square.
>
> *Example.*—3 is a nonsquare number, which multiplied by 1, which is a square, makes 3, and by adding unity makes 4, which is a square.
>
> The same 3, multiplied by 16, which is a square, makes 48, and with unity added makes 49, which is a square.
>
> There is an infinity of such squares which when multiplied by 3 with unity added likewise make a square number.
>
> I demand a general rule,—given a nonsquare number, find squares which multiplied by the given number, and with unity added, make squares.
>
> What is for example the smallest square which, multiplied by 61 with unity added, makes a square?
>
> Moreover, what is the smallest square which, when multiplied by 109 and with unity added, makes a square?
>
> If you do not give me the general solution, then give the particular solution for these two numbers, which I have chosen small in order not to give too much difficulty.
>
> After I have received your reply, I will propose another matter. It goes without saying that my proposition is to find integers which satisfy the question, for in the case of fractions the lowest type of arithmetician could find the solution.

Connected with this problem are a number of others, assembled by Fermat's friend Jacques de Billy (1602–1669), a Jesuit teacher of mathematics in Dijon, in his *Doctrinae analyticae inventum novum* (ed. S. Fermat; Toulouse 1670), translated in Fermat, *Oeuvres*, III, 325–398. They begin with the Diophantine problem (called a double equation), to make both $2x + 12$ and $2x + 5$ squares (answer $x = 2$). Part III (p. 376) begins (we change to modern notation):

> On the procedure for obtaining an infinite number of solutions which give square or cubic values to expressions in which enter more than three terms of different degrees.
>
> 1. I shall discuss here in particular expressions which contain the five terms in x^4, x^3, x^2, x, and the constant, but I also wish to discuss expressions with four terms which may be all positive [true], or mixed with negative [false] terms. We wish to give these expressions square values (in the case of five terms), or

cubic ones (in the case of four terms), and this in an infinity of ways. In general we must say that for the square value at least the coefficient of the term in x^4 or the constant term must be a square; as to the cubic values, the coefficient of x^3 or the constant term must be a cube.

Applied to making $x^4 + 4x^3 + 6x^2 + 2x + 7$ a square, De Billy writes $(x^2 + 2x + 1)^2 = x^4 + 4x^3 + 6x^2 + 4x + 1$, which, set equal to the given form, gives $x = 3$.

In the case of $x^4 + 4x^3 + 10x^2 + 20x + 1$ De Billy equates this to $(1 + 10x - 45x^2)^2$, and gets $x = \frac{113}{253}$, then he equates it to $(x^2 + 2x - 1)^2$, and gets $x = -3$, and so on.

Then, by substituting for x the value $x + x_0$, where x is a "primitive" solution, for example $x_0 = -3$, or $x_0 = -4$, and repeating the process, he obtains new solutions. For $x \to x - 3$ he requires that $x^4 - 8x^3 + 28x^2 - 40x + 4$ be a square, which gives $x = \frac{7}{2}$; hence $x = \frac{1}{2}$ is a solution of the original equation. Here he turned a "false" solution into a "true" one. This process can be repeated.

It was from these problems by Fermat that Euler, in the paper of 1732/33, started his research on the "Pell" equation.