Barrow goes on to give more examples and Art. 19 arrives again at the fundamental theorem, now in the form converse to that given in Lecture X, Art. 11: if the curve AMB is given by $z = f(x)$, and the curve KZL by $y = f_1(x)$, $z\,dx/dz : z = R : y$, then $\int y\,dx = R\int dz$, or $\int y\,dx = Rz$.

We list here some of Barrow's notations which we have modified: $A \sqsubset B$, A is greater than B; $A \sqsupset B$, A is less than B; $A\,.\,B :: C\,.\,D$, $A : B = C : D$; Aq, the square of A, for instance, in Lecture X, Art. 13, square on DG is written DGq; Ac or A cub, the cube of A; $DHqq$ the fourth power of DH. We have kept his symbol of multiplication, $A \times B$.

We end with a word of caution. Despite the fact that, in order to understand these seventeenth-century mathematicians, we are inclined to translate their reasoning into the notation and language with which we are familiar, we must constantly be aware that our point of view is not equivalent to theirs. They saw geometric theorems in the sense of Euclid, where we see operations and calculating processes. At the same time, just because these mathematicians applied their geometric notions in an attempt to transcend the static character of classical mathematics, their geometric thought has a richness that may easily escape observation in the modern transcription. If we were to rewrite Euclid in the notation of analytic geometry we would obtain a body of knowledge with a character different from that of Euclid and, despite all the advantages that the algebraic computations would bring, we would lose some of the more subtle and esthetic qualities of Euclid.

15 HUYGENS. EVOLUTES AND INVOLUTES

The search for reliable clocks, a necessity for scientific navigation and geography as well as for theoretical astronomy, led Christiaan Huygens (1629–1695), a Dutch patrician and a founding member of the French Academy of Sciences (1666), to the invention of the pendulum clock (the idea of which seems to have already occurred to Galilei). Huygens described this invention in the *Horologium oscillatorium* (Paris, 1673; reprinted, with French translation, in *Oeuvres complètes de Christiaan Huygens*, XVIII, 68–368). This book, in its five parts, contains a number of important discoveries in mechanics and mathematics, so that, with the books of Cavalieri and Wallis (see Selections IV.5, 6, 13), it is a landmark on the path that led to the invention of the calculus.

After describing his pendulum clock in Part I, Huygens deals in Part II with "The fall of heavy bodies and their cycloidal movement." Here we find a theory of the cycloid and, based on it, the following theorem on a heavy point moving on a cycloid in a field of gravity:

Proposition XXV. On a cycloid with a vertical axis whose vertex is below, the times of descent in which a mobile point, starting from rest at an arbitrary point of the curve, reaches the lowest point, are all equal, and have to the times of the vertical fall along the total axis of the cycloid a ratio equal to that of the semicircumference of a circle to that of the diameter [in our terms, as $\pi : 2$].

In other words, the cycloid is a *tautochrone*. From this theorem Huygens obtains the tautochronic pendulum, which has a period independent of its amplitude. This property of the cycloid leads him to the discovery that the evolute of the cycloid is also a cycloid, and then, in Part III of the book, to the general theory of evolutes and involutes of plane curves.

The *Horologium oscillatorium* can also be studied in a German translation in Ostwald's *Klassiker*, No. 192 (Engelmann, Leipzig, 1913).

Here follows a translation of the beginning of Part III, entitled "On the evolution and dimension of curved lines."

Definition I. A curve is said to be curved [*inflexa*] to one side, if all its tangents touch it just on that side. If it has some parts straight, then these, continued at both ends, are themselves regarded as tangents.

Definition II. When two curves of this kind pass through the same point, and when the convexity of one is directed toward the concavity of the other, as the curves ABC and ADE [Fig. 1], then we shall call both "concave [*cavae*] to the same side."

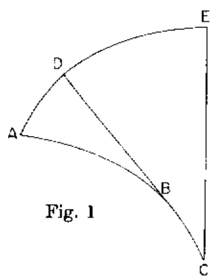


Fig. 1

Definition III. When we consider that a thread or flexible line is laid along a curve concave to one side, and when we remove one end from it while the other end of the thread stays on the curve in such a way that the developed part remains taut, then this end of the thread will clearly describe another curve; this curve is called an involute [*descripta ex evolutione*].[1]

Definition IV. The curve, however, along which the thread has been laid may be called the evolute [*evoluta*]. In the figure ABC is the evolute, ADE the involute of ABC, for if the end of the thread has come from A to D, then the straight part DB of the thread will be taut, while the other part BC still lies along the curve. It is clear that DB is tangent to the evolute at B.

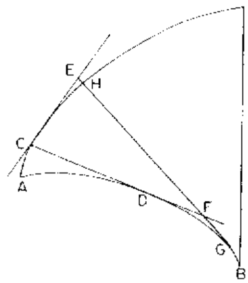
Proposition I. Every tangent of the evolute intersects the involute at right angles:

Let AB [Fig. 2] be the evolute, AH its involute. Let the straight line FDC, tangent to curve ADB at D, intersect the curve ACH at C. I claim that it

[1] The term *involute*, a curious English construction for the more natural "evolvent" (as in German: *Evolute, Evolvente*; in French: *développée, développante*) appears first in Charles Hutton's *Mathematical dictionary* (London, 1796), according to the *Oxford English Dictionary*. This gives for the first English appearance of the term *evolute* 1730–36.

intersects the curve at right angles, that is, if we construct on CD the perpendicular CE, then this line should touch the curve ACH at C. Indeed, since the straight line DC is tangent to the evolute at D, it clearly represents the position of the thread at the moment when its end has come to C. When therefore we prove that the thread while describing the whole curve ACH can reach the line CE only at the point C, we shall have proved that CE is tangent to the curve ACH at the point C.

Fig. 2



Let us take on AC another point H different from C, and let us consider first the case in which H is farther removed than C from the starting point A of the evolution. Let the free part of the thread have the position HG, when its end is at H. The line HG is therefore tangent to the curve AB at G. While the end of the thread describes the arc CH, the thread evolves itself away from arc DG. Hence CD will intersect the line HG if extended beyond D; say at F. Let GH intersect the line CE at E. We then have[2]

$$DF + FG > DG,$$

whether DG be a straight or a curved line. If we add to both sides the straight segment DC, then we obtain

$$CF + FG > CD + DG.$$

In connection with the evolution we have

$$CD + DG = HG.$$

Hence the sum $CG + FG$ will also be $> HG$, and if we subtract from both sides the segment FG, then we find that

$$CF > HF.$$

But we have

$$FE > FC,$$

[2] Huygens does not use the Harriot symbol $>$, but uses words: "Quia igitur duae simul DF, FG, majores sunt quam DG."

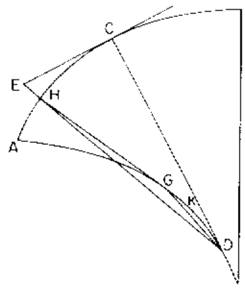
since in the triangle FCE the angle C is right. Hence we have a fortiori

$$FE > FH.$$

From this it follows that the thread on this side of the point C no longer intersects the line CE.

Now let the point H [Fig. 3] be closer to the starting point A than the point C. Let HG be the position of the thread at the moment when its end is at H. Let

Fig. 3



us draw the lines DG and DH, of which the last one meets the straight line CE at E. It is clear that the straight line DG cannot be on the continuation of HG, and that HGD is therefore a triangle. Now, since

$$DG \leqslant DKG,$$

the sign $=$ holding for the case where the part DG of the evolute is straight, we will find, adding GH on both sides, that

$$DG + GH \leqslant DKG + GH,$$

or

$$DG + GH \leqslant DC.$$

But

$$DH < DG + GH,$$

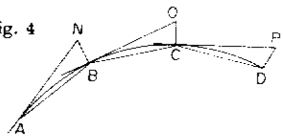
hence DH is a fortiori $< DC$. But $DE > DC$, since in triangle DCE the angle C is right. Hence DH is much more $< DE$. The point H, the end of the thread GH, is therefore situated inside the angle DCE. From this it follows that between A and C the end of H never gets as far as CE. Hence CE touches the curve AC at C, so that DC, to which CE has been constructed as a perpendicular, cuts the curve at right angles. Q.E.D.[3]

[3] We see that Huygens proves that a line is tangent to the curve by using the method of the ancients; see our commentary to Selections IV.8, 10.

We give the next three propositions without the proof, which is similar in character to that given above for Proposition I.

Proposition II. Every curved line segment, concave in one direction, as ABD [Fig. 4], can be divided in so many parts that if we draw the chords that subtend every one of the arcs, as AB, BC, and CD, and then draw the tangents AN,

Fig. 4

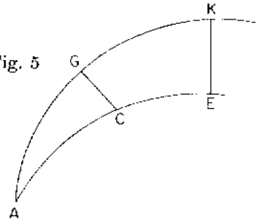


BO, CP from every one of the points of division and also from the end of the curve, till each of them meets the normal to the curve at the next point of division (BN, CO, DP are the normals), then every chord will have to the corresponding normal (AB to BN, BC to CO, CD to PD) a ratio superior to any given ratio.[4]

Proposition III. Two curved lines curved both in the same direction and concave in that same direction cannot issue from the same point in such a mutual position that every line normal to the one is also normal to the other.

Huygens proves that if ACE and AGK [Fig. 5] are such curves, and KE is a common normal, then the proposition that all normals at the points G of AGK are also normals to ACE leads to an absurdity.

Fig. 5



Proposition IV. If from a point pass two curved lines curved both in the same direction and concave in the same direction, and in such a mutual position that the tangents to the one of them meet those of the other at right angles, then this other curve will be the evolute of the first from the common point on.

[4] In our words: the segments on the normals are of higher order of infinity than the chords. See Selection V.5.

Proposition V. If a straight line touches a cycloid at its vertex and we construct on this straight line as base another cycloid, similar and equal to the first, then, beginning at the vertex mentioned, an arbitrary tangent to the inferior cycloid will be normal to the superior cycloid.

Let us suppose that the straight line AG [Fig. 6] touches the cycloid at its vertex A and that on this line as base is constructed another cycloid AEF with

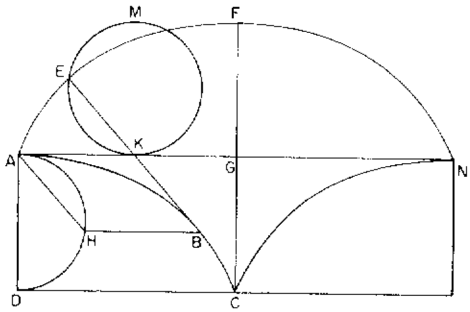


Fig. 6

vertex F. Let BK be a tangent to the cycloid ABC. I claim that this tangent, continued to the cycloid AEF, will meet it at right angles. Indeed, let us describe about AD, the axis of the cycloid ABC, the generating circle AHD, which intersects BH, parallel to the base, at H. Let us draw the line HA. Since BK is tangent to the cycloid at B, it is parallel to this line HA, hence $AHBK$ is a parallelogram and $AK = HB$, that is, equal to the arc AH.[5] Let us now describe the circle KM, equal to the generating circle AHD, tangent to the base AG at K and intersecting the continued line BK at E. Since BKE is parallel to AH, and hence $EKA = KAH$, it is clear that the continued line BK cuts from the circle KM an arc equal to the arc which AH cuts from the circle AHD. The arc KE is therefore equal to the arc AH, that is, to the line HB, hence to the line KA. But it follows from this equality, from a property of the cycloid (since the generating circle MK is tangent to line AG at K), that the point which describes the cycloid [AEF] has passed through E. The line KE therefore meets the cycloid at E at right angles, that is, KE is no other line than the continuation of BK. Q.E.D.

Proposition VI. By the evolution of a half-cycloid, beginning at the summit, another half-cycloid is described equal and similar to the first, whose base coincides with the straight line that is tangent to the cycloid evolved at its vertex.

In the following propositions of Part III Huygens investigates many other evolutes, notably those of conic sections, and uses this information for some computations of length

[5] In this proof Huygens uses several properties of the cycloid that he has established in Part II of his book. See also Selection IV.10.

and area. This leads to the general theorem on the construction of evolutes for "geometrical" curves.[6]

Part IV of Huygens' book, "On centers of oscillation," contains the theory of the oscillating bodies. The short Part V deals with centrifugal force and another clock construction.

[6] The establishment of this theorem is the beginning of a series of investigations on curves that are congruent or similar to their evolutes. The search leads to logarithmic spirals, epicycloids, and hypocycloids. See *Oeuvres complètes de Christiaan Huygens*, XVIII, 40–41, and C. A. Crommelin and W. van der Woude, *Simon Stevin 30* (1954), 17–24. As to "geometric curves," Descartes, in the second book of his *Géométrie*, called curves "geometric" if they "admit of precise and exact measurement," so that all their points must bear a definite relation to all points of a straight line, a relation to be expressed by means of a single equation, which then can be of different degrees (see Selection III.4). For such curves Huygens expresses, in geometric form, the formula for the radius of curvature which later Jakob Bernoulli would express by $z = ds^3 : dy\, d\, dx$, a formula equivalent to the one familiar to us in our calculus texts; see "Curvatura laminae elasticae ... Radii circulorum osculatium in terminis simplicissimis exhibiti ...," *Acta Eruditorum* (June 1694), 262–266 (*Opera* II, 576–600). See further Selection V.23.