Hence $\log \sin 3' = \log 8727 = 70439560,$

$\log \sin 30° 1' = \log 5002519 = 6926432,$

$\log \sin 45° = \log 7071068 = 3465735;$ (half of log sin 30°, Art. 56),

also $\log \sin 90° = \log 10000000 = 0.$

5 PASCAL. THE PASCAL TRIANGLE

The so-called Pascal triangle appears in a treatise by Blaise Pascal (1623–1662), published posthumously under the title *Traité du triangle arithmétique, avec quelques autres petits traités sur la même manière* (Paris, 1665). This treatise is important not only because of its careful examination of the properties of the binomial coefficients, but also because of their application to problems in games of chance. At one place Pascal expresses with clarity the principle of complete induction.

The Pascal triangle appears for the first time (so far as we know at present) in a book of 1261 written by Yang Hui, one of the mathematicians of the Sung dynasty in China.[1] The properties of binomial coefficients were discussed by the Persian mathematician Jamshid Al-Kāshī in his *Key to arithmetic* of c. 1425.[2] Both in China and in Persia the knowledge of these properties may be much older. This knowledge was shared by some of the Renaissance mathematicians, and we see Pascal's triangle on the title page of Peter Apian's German arithmetic of 1527. After this we find the triangle and the properties of binomial coefficients in several other authors.[3]

Pascal wrote his treatise probably by the end of 1654. It can be found in the *Oeuvres*, ed. L. Brunschvicg and P. Boutroux, III (Hachette, Paris, 1909), 456 seq., and in other editions of Pascal's work. A paraphrase of certain theorems can be found in H. Meschkowski, *Ways of thought of great mathematicians* (Holden-Day, San Francisco, 1964), 36–43.

TREATISE ON THE ARITHMETIC TRIANGLE

I designate as the arithmetic triangle a figure of which the construction is as follows [Fig. 1]. Through an arbitrary point G I draw 2 lines perpendicular to each other, GV and $G\zeta$, on each of which I take as many equal and continuous parts as I like, beginning at G, which I call 1, 2, 3, 4, etc., and these numbers are the indices [*exposans*] of the divisions of the lines.

Then I join the points of the first division, which are on each of the two lines, by another line that forms a triangle of which this line is the *base*.

I also join the two points of the second division by another line that forms a second triangle of which this line is the *base*.

[1] J. Needham, *Science and civilisation in China*, III (Cambridge University Press, New York, 1959), 135.

[2] Russian translation by B. A. Rozenfel'd (Gos. Izdat, Moscow, 1956); see also Selection I.3, footnote 1.

[3] Smith, *History of mathematics*, II, 508–512. See also our Selection II.9 (Girard).

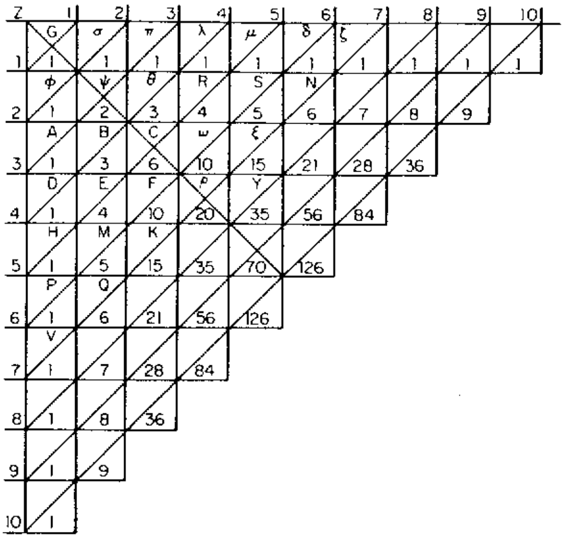

Fig. 1

And joining in this way all the division points which have the same indices I form with them as many *triangles* and *bases*.

I draw through every one of the division points lines parallel to the sides, and these by their intersections form small squares which I call cells [*cellules*].

And the cells that are between two parallels that run from left to right are called *cells of the same parallel rank*, such as the cells G, σ, π, etc., or φ, ψ, θ, etc.

And those that are between two lines that run from the top downward are called *cells of the same perpendicular rank*, such as the cells G, φ, A, D, etc. and these: σ, ψ, B, etc.

And those that the same base traverses diagonally are called *cells of the same base*, such as the following: D, B, θ, λ, and these: A, ψ, π.

The cells of the same base that are equally distant from their extremities are called *reciprocal*, such as these: E, R and B, θ, because the index of the parallel rank of the one is the same as the index of the perpendicular rank of the other, as appears in the example, where E is in the second perpendicular and in the fourth parallel rank, and R is in the second parallel and in the fourth perpendicular rank, reciprocally. It is easy enough to show that those which have their indices reciprocally equal are in the same base and equally distant from its extremities.

It is also quite easy to demonstrate that the index of the perpendicular rank of any cell whatsoever, added to the index of its parallel rank, exceeds the index of its base by unity.

For example, the cell F is in the third perpendicular rank and in the fourth parallel one, and in the sixth base, and its two indices of the ranks $3 + 4$ exceed the index 6 of the base by unity, which results from the fact that the two sides of the triangle are divided into an equal number of parts, but that is rather understood than demonstrated.

The following remark is of the same nature: that every base contains one cell more than the preceding one, and every one contains as many cells as its index has units; the second base $\varphi\sigma$, for instance, has two cells, the third $A\psi\pi$ has three of them, etc.

We now place numbers in each cell and this is done in the following way: the number of the first cell which is in the right angle is arbitrary, but once it has been placed all the other numbers are determined, and for this reason it is called the *generator* of the triangle. And every one of the other numbers is specified by this sole rule:

The number of each cell is equal to that of the cell preceding it in its perpendicular rank plus that of the cell which precedes it in its parallel rank. For instance, the cell F, that is, the number of the cell F, is equal to cell C plus cell E, and so the others.

From this many consequences can be drawn. Here are the most important ones, where I consider the triangles whose generator is unity, but what can be said about them will also apply to the others.

FIRST CONSEQUENCE

In every arithmetic triangle all the cells of the first parallel rank and of the first perpendicular rank are equal to the generator.

Indeed, by the construction of the triangle, every cell is equal to the cell which precedes it in its perpendicular rank plus the cell that precedes it in its parallel rank. Now, the cells of the first parallel rank have no cells which precede them in their perpendicular ranks, nor have those of the first perpendicular rank any in their parallel ranks: hence they are all equal to each other and to the generating first number.

And so φ is equal to G + zero, that is, φ is equal to G.

And so A is equal to φ + zero, that is, φ.

And so σ is equal to G + zero, and π equal to σ + zero.

And so the others.

Using a more modern notation, in which we call P_l^k the cell of parallel rank l and vertical rank k, so that

$$P_l^k = \frac{(k + l - 2)!}{(k - 1)!(l - 1)!},$$

we can write the next "consequences" as follows:

2. $$P_l^k = \sum_{i=1}^{k} P_{l-1}^i; \quad \text{e.g., } \omega = R + \theta + \psi + \varphi;$$

3. $$P_l^k = \sum_{i=1}^{k} P_i^{k-1}; \quad \text{e.g., } C = B + \psi + \sigma;$$

4. $$P_l^k - 1 = \sum_{i=1}^{k-1} \sum_{j=1}^{l-1} P_j^i; \quad \text{e.g., } \xi - g = R + \theta + \psi + \varphi + \lambda + \pi + \sigma + G,$$

where $g = 1$, the generator;

5. $$P_l^k = P_k^l; \quad \text{e.g., } \varphi = \sigma = G, \pi = A = G, D = \lambda = G.$$

6. All $P_l^k = P_k^l$, k fixed; e.g., $\sigma\psi BEM\varphi$ is equal to $\varphi\psi\theta RSN$;

7. $$\sum_{l,k=1,\ldots,n} P_l^k = 2 \sum_{i,j=1,\ldots,n-1} P_j^i, \quad k + l = \text{fixed number} = a, \quad i + j = a - 1;$$

e.g., $D + \lambda + B + \theta = 2A + 2\psi + 2\pi$;

8. $$\sum_{l,k=1,\ldots,n} P_l^k = 2^{n-2}, \quad k + l = n;$$

9. $$1 + 2 + \cdots + 2^n = 2^{n+1} - 1;$$

10. $$\sum_{l=n}^{p} P_l^k = 2 \sum_{i=n-1}^{p-1} P_j^i + P_{n-1}^p \quad [\text{e.g., } P_4^1 + P_3^2 + P_2^3 = 2(P_3^1 + P_2^2) + P_4^3],$$

$k + l = n, \quad i + j = n - 1, \quad p = n - 2$; e.g., $D + B + \theta = 2A + 2\varphi + \pi$;

11. $$P_l^l = 2P_l^{l-1} = 2P_{l-1}^l; \quad \text{e.g., } C = \theta + B = 2B.$$

TWELFTH CONSEQUENCE

In every arithmetic triangle, if two cells are contiguous in the same base, the upper is to the lower as the number of cells from the upper to the top of the base is to the number of those from the lower to the bottom, inclusive.

Let the two contiguous cells, arbitrarily chosen on the same base, be E, C; then I say that

E	is to	C	as	2	is to	3
lower one		upper one		because there are two cells between E and the first, namely E, H;		because there are three cells between C and the top, namely C, R, μ.

Although this proposition has an infinite number of cases I shall give for it a very short demonstration by supposing two lemmas:

The first one, evident in itself, is that this proportion occurs in the second base; because it is clear enough that φ is to σ as 1 is to 1.

The second one is that if this proposition is true in an arbitrary base, it will necessarily be true in the next base. From which it is clear that it will necessarily be true in all bases, because it is true in the second base because of the first lemma; hence by means of the second lemma it is true in the third base, hence in the fourth base, and so on to infinity.[4]

It is therefore necessary to demonstrate only the second lemma, and this can be done in the following way. Let this proportion be true in an arbitrary base, as in the fourth one D, that is, if D is to B as 1 is to 3, and B to θ as 2 to 2, and θ to λ as 3 to 1, etc., then I say that the same proportion will be true in the next base, $H\mu$, and that, for example, E is to C as 2 is to 3.

Indeed, D is to B as 1 is to 3, by hypothesis.

Hence $\underbrace{D + B}_{E}$ is to B as $\underbrace{1 + 3}_{4}$ is to 3.

In the same way: B is to θ as 2 is to 2, by hypothesis.

Hence $\underbrace{B + \theta}_{C}$ is to B as $\underbrace{2 + 2}_{4}$ is to 2.

But B is to E as 3 is to 4.

Hence, by the double proportion,[5] C is to E as 3 is to 2. Q.E.D.

The proof can be given in the same way in all the other cases, since this proof is founded only on the fact that this proportion is true in the preceding base, and that every cell is equal to its preceding one plus the one above it, which is true in all cases.[6]

There follow more "consequences," numbered 13–19.[7] The article ends with a "Problem":

Given the indices of the perpendicular and of the parallel rank of a cell, to find the number of the cell, without using the arithmetic triangle.

[4] This seems to be the first satisfactory statement of the principle of complete induction. See H. Freudenthal, "Zur Geschichte der vollständigen Induktion," *Archives Internationales des Sciences 22* (1953), 17–37.

[5] The text has "proportion troublée," probably a misprint for "proportion doublée."

[6] The meaning of this is as follows. Given

$$P_k^l : P_{k-1}^{l+1} = \frac{l}{k-1} \quad \text{(in base } k + l - 1\text{)}.$$

But

$$P_k^l + P_{k+1}^{l+1} = P_{k-1}^{l+1} \quad \text{(rule of formation of the triangle);}$$

hence

$$P_k^{l+1} : P_{k-1}^{l+1} = \frac{l + k - 1}{k - 1},$$

$$P_{k-1}^{l+1} : P_{k-2}^{l+2} = \frac{l + 1}{k - 2},$$

$$P_{k-1}^{l+2} : P_{k-1}^{l+1} = \frac{l + k - 1}{l + 1};$$

hence

$$P_k^{l+1} : P_{k-1}^{l+2} = \frac{l + 1}{k - 1} \quad \text{(in base } k + l\text{)}.$$

[7] For example, consequence 17 states that

$$\sum_{i=1}^{k} P_i^l : \sum_{j=1}^{l} P_k^j = k : l, \quad \text{e.g., } (B + \psi + \sigma) : (B + A) = 3 : 2.$$

These consequences can all be found in the translation of Pascal's paper in Smith, *Source book*, pp. 74–75.

For example, let it be proposed to find the number of the cell ξ of the fifth perpendicular rank and of the third parallel rank.

Having taken all the numbers that precede the index of the perpendicular rank 5, that is, 1, 2, 3, 4, take as many natural numbers beginning with the index of the parallel rank 3, that is, 3, 4, 5, 6.

Now multiply the first numbers into each other, and let the product be 24. Multiply the other numbers into each other, and let the product be 360, which divided by the other product 24, gives 15 as the quotient. This quotient is the desired number.

Indeed, ξ is to the first number of its base V in composed ratio of all the ratios of the cells among themselves, that is,

$$\xi \text{ is to } V \text{ in composed ratio of } \underbrace{\xi \text{ to } \rho}_{3 \text{ to } 4} + \underbrace{\rho \text{ to } K}_{4 \text{ to } 3} + \underbrace{K \text{ to } Q}_{5 \text{ to } 2} + \underbrace{Q \text{ to } V}_{6 \text{ to } 1},$$

or by the twelfth consequence:

$$\xi \text{ is to } V \text{ as 3 into 4 into 4 into 5 into 6 into 4 into 3 into 2 into 1},$$

But V is unity; hence ξ is the quotient of the division of the product of 3 into 4 into 5 into 6 by the product of 4 into 3 into 2 into 1.[8]

Note. If the generator were not unity we should have to multiply the quotient by the generator.

This paper is followed by several others, in which the Pascal triangle is applied.[9] First it is used to sum the arithmetical sequences of different orders 1, 2, 3, 4, etc.; 1, 3, 6, 10, etc., 1, 4, 10, 20, ... (these sequences are called "numbers of the first, second, etc. order" [*ordres numériques*], then to the solution of certain games of chance, to the finding of combinations, to the raising of binomials to different powers, to the summation of the squares, cubes, etc., of the terms of an arithmetical series, etc., and to the proof that (in our present notation) $\int_0^a x^p\,dx = \frac{a^{p+1}}{p+1}$, p a positive integer. On this integral see Selection IV.6.

6 FERMAT. TWO FERMAT THEOREMS AND FERMAT NUMBERS

Pierre de Fermat (1601–1665) was a lawyer attached as councilor to the provincial parliament (that is, law court) of Toulouse. Of his contributions to geometry and calculus we speak in Selections III.3 and IV.7, 8. He was the first to take up seriously the challenge offered in number theory by the *Arithmetica* of Diophantus, first made fully available in the original Greek of A.D. *c.* 250 by Claude Bachet in 1621, together with a Latin translation. Fermat communicated his results in letters to his friends or kept them to himself in notes,

[8] This means that $P^l_k = \frac{l(l+1)\cdots(l+k-2)}{1\cdot 2\cdots(k-1)} = C^{l+k-2}_{k-1}$; hence $C^n_p = P^{n-p+1}_{p+1}$, where $C^n_p = \frac{m!}{p!(n-p)!}$, the number of combinations of n elements in groups of p.

[9] Some of this is translated in Smith, *Source book*, pp. 76–79.