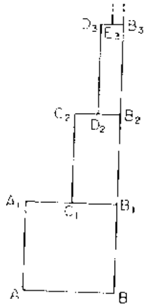


Fig. 9

10 ROBERVAL. THE CYCLOID

The cycloid has been traced back to the French theologian and mathematician Charles Bouvelles (*c.* 1470–*c.* 1553). Galilei was attracted by it, and wrote in a letter to Cavalieri of February 24, 1640 (*Opere*, Edizione nazionale (Barbera, Florence, 1890–1909), XVIII, 153–154): "More than 50 years ago this curved line came to my mind and I wanted to describe it, admiring it because of its most gracious curvature, adaptable to the arches of a bridge. I made several tentative calculations on it and on the space comprised between it and its chord, in order to demonstrate some property. And it seemed at first that such space may be three times the circle which it describes, but it was not that." Galilei gave the curve its name.

About 1630 Father Marin Mersenne (1588–1648), a correspondent of Descartes, Fermat, and many other mathematicians, suggested the cycloid as a test curve for the different methods of dealing with infinitesimals. It soon became one of the most discussed curves of the period, the discussion occasionally leading to acrimonious remarks, so that the curve has been compared to an apple of discord or called the Helen of the geometers. Among those who took up the challenge of Mersenne was Gilles Personne de Roberval (1602–1675), a professor of mathematics in Paris at the Collège du Roi (now Collège de France). From his *Traité des indivisibles* (1634; first published Paris, 1693; reprinted Paris, 1730; Amsterdam, 1736) we present here a section on the cycloid, translated rather freely (the original is somewhat prolix) by E. Walker in *A study of the Traité des Indivisibles* (Teachers College, New York, 1932). It shows how Roberval handled indivisibles,[1] and how he introduced the so-called companion of the cycloid, that is, the sine curve, which was long known under this name, even in the days of Euler. Roberval usually called the cycloid a *roulette*, a custom followed by Pascal; another name was *trochoid* (after Greek *trochos*, wheel). We have, with Walker, used the now customary term cycloid. The interest in this curve was also connected with the age-old speculation concerning the *rota Aristotelis* (see Selection IV.3).

On Roberval see further L. Auger, *Un savant méconnu, G. P. de Roberval* (Blanchard, Paris, 1962). On his mathematics see also C. B. Boyer, *The history of the calculus* (Dover,

[1] It is clear that Roberval, like Cavalieri, uses the method of indivisibles, of which he may have been an independent discoverer (Walker, *A study of the Traité des indivisibles*, 15, 142), but his view was somewhat different. He made clear in his *Traité* that the phrase "the infinite number of points" stands for the infinity of little lines which make up the whole line; see Boyer, *History of the calculus*, 141–142.

New York, 1949). On the cycloid see E. A. Whitman, "Some historical notes on the cycloid," *American Mathematical Monthly 50* (1943), 309–315.

We follow here, with some modifications, the text in Walker, 174–177, 219–222, corresponding to pp. 209ff of the 1736 edition of Roberval's *Traité*. The Walker version also introduces some modern symbolism, and the division into Propositions 1, 2, . . . is Walker's. Her book also contains translations and paraphrases of other sections of the *Traité*.

To Generate the Cycloid. Let the diameter AB [Fig. 1] of the circle $AEGB$ move along the tangent AC, always remaining parallel to its original position, until it takes the position CD, and let AC be equal to the semicircle AGB. At the same time, let the point A move on the semicircle AGB, in such a way that the speed of AB along AC may be equal to the speed of A along the semicircle AGB. Then, when AB has reached the position CD, the point A will have reached the position D. The point A is carried along by two motions—its own on the semicircle $AEGB$, and that of the diameter along AC. The path of the point A, due to these two motions, is the half cycloid $A \cdots D$, the second half being symmetrical with the first.

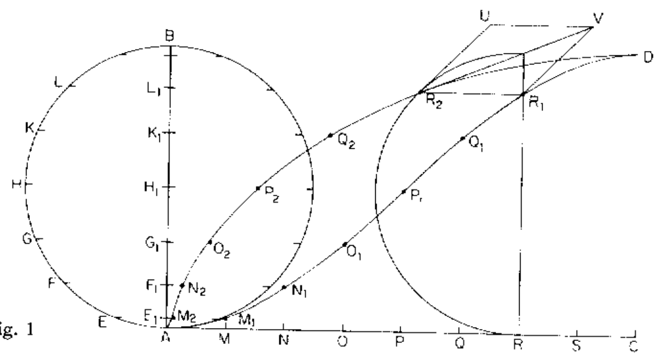


Fig. 1

The Nature of the Cycloid. Let the line AC and the semicircle AGB be divided into an infinite number of parts such that arc AE = arc $EF = \cdots$

$$= \text{line } AM = \text{line } MN = \text{line } NO = \cdots.$$

Draw the sine EE_1 perpendicular to the diameter AB, and the versed sine AE_1 is the altitude of A when it has come to E. Similarly draw FF_1, GG_1, etc.

Let MM_1 be parallel and equal to AE_1, NN_1 parallel and equal to AF_1, etc. Let M_1M_2 be parallel to AC and equal to EE_1, N_1N_2 parallel to AB and equal to FF_1, etc. [Roberval's notation for $M_1, N_1, \ldots$ is 1, 2, . . .; for $M_2, N_2, \ldots$ is 8, 9, ...]

When the diameter has reached the point M, the point A will have reached the position E, the distance of A above AC will be $MM_1 = AE_1$, and the distance of A from the diameter AB will be $EE_1 = M_1M_2$, hence when the diameter is at M the point A is at M_2. In the same way, when the diameter is at N the point A is at N_2, etc. We thus get two curves, one $AM_2N_2 \cdots R_2D$, and the other $AM_1N_1 \cdots R_1D$. The first of these is the path of the point A, which is the first half of the cycloid.

The Companion of the Cycloid. The curve drawn through the points $AM_1N_1 \cdots R_1D$, is known as the companion of the cycloid.[2]

Proposition 1. *The area of the figure included between the cycloid and the companion of the cycloid is equal to the area of half of the generating circle.*

Proof. Within the figure $AM_2N_2 \cdots D \cdots N_1M_1 \cdots A$ we have $M_1M_2 = EE_1$, $N_1N_2 = FF_1$, $O_1O_2 = GG_1$, etc.

Now M_1M_2, N_1N_2, O_1O_2 divide this figure into strips whose altitudes are $AE_1, E_1F_1, F_1G_1, \ldots$, while $EE_1, FF_1, GG_1, \ldots$ divide the semicircle AHB into strips having the same altitudes. Hence the corresponding infinitesimal strips are equal. Therefore the area of the figure $AM_2N_2 \cdots D \cdots N_1M_1 \cdots A$ is equal to the area of the semicircle AHB.[3]

Proposition 2. *The area of the figure included between the cycloid and its base is equal to three times the area of the generating circle.*

Proof. The companion of the cycloid, the curve $AM_1N_1 \cdots D$, bisects the parallelogram $ABCD$, since to each line in $ACDM_1$ there corresponds an equal line in $ABDM_1$.

Therefore the area of $ACDM_1$ = $\frac{1}{2}$ the area of $ABCD$
= $\frac{1}{2}$,, ,, ,, 2·circle AGB
= ,, ,, ,, circle AGB.

Therefore the area of $ACDM_2 = ACDM_1 + AM_2 \cdots D \cdots M_1$
= circle AGB + $\frac{1}{2}$ circle AGB
= $\frac{3}{2}$ circle AGB.

Doubling, the area between the whole cycloid and its base is equal to three times the area of the generating circle.

Proposition 3. *To construct a tangent to the cycloid.*

Construction. Let R_2 be the given point at which the tangent is to be drawn. Draw R_2R_1 parallel to AC. Draw R_2U tangent to the generating circle RR_2 and make $R_2U = R_2R_1$. Complete the parallelogram R_2UVR_1, and draw the diagonal R_2V. Then R_2V is the required tangent.

Proof. The direction of the motion of the point R_2 which is due to the motion of AB along AC is R_2R_1; the direction of the motion of the point R_2 which is due to the motion of the point A on the circumference is R_2U, and since these motions are always equal, it follows that R_2R_1 must equal R_2U. Therefore R_2V is the tangent to the cycloid at R_2, since it is the resultant of the two motions.

[2] The "companion of the cycloid" is a sine curve. If AC is taken as the X-axis, AB as the Y-axis, its equation is, in our notation, $y = 1 - \cos x$.

[3] When arc $AE = \varphi$ [radius $R = 1$], then the equation of the cycloid is $x = \varphi - \sin\varphi$, $y = 1 - \cos\varphi$, and the area $AM_2DM_1A = \int_0^\pi (\varphi - \varphi + \sin\varphi)^2\, dy = \int_0^\pi \sin^2\varphi\, d\varphi = \pi/2$.

Addendum. If, instead of being equal, the magnitudes of the two motions had been in some other ratio, the parallelogram would have been constructed with its sides in that ratio.[4]

The next 22 Propositions deal with a number of area computations and other integrations. Curves discussed include the parabola, the limaçon of Pascal (which Roberval calls the conchoid of the circle), ring surfaces, the hyperboloid of revolution (which Roberval calls the hyperbolic conoid), cones, spheroids, the conchoid of Nicomedes, and the curve introduced as follows, which is known as the hippopede of Eudoxus.[5]

Proposition 26. *On the surface of a right cylinder draw a line enclosing an area equal to the area of a given square, and that with a single stroke of the compasses.*

Construction. Let AB [Fig. 2] be the side of the given square. Bisect AB at C. Describe a circle FME whose diameter FE is equal to AC. Construct a right cylinder whose midsection is the circle FME, and whose altitude is equal at

Fig. 2

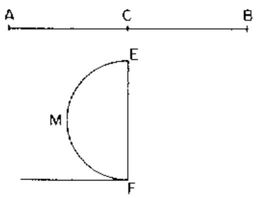
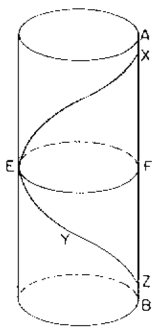


Fig. 3



least to $2FE$. With F as a fixed point, and with an opening of the compasses equal to FE, draw a closed curve XYZ on the cylinder [Fig. 3]. Then XYZ is the required curve enclosing an area equal to the square on AB.

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[4] The tangent construction uses kinematic concepts and is related to the method of Archimedes in his book *On spirals*; see T. L. Heath, *The works of Archimedes* (Cambridge University Press, Cambridge, England, 1897; reprint, Dover, New York), 151ff, esp. Props. 16–20. While Roberval used Greek methods to find tangents, his contemporary Fermat was laying the foundations of the present method, based on the derivative (see Selection IV.8).

[5] This curve, which plays a role in the planetary model constructed by Eudoxus (fourth century B.C.), was one of the curves discussed by Mersenne, Fermat, Roberval, and other mathematicians of their day, including the Toulouse mathematician Antoine de Lalouvère, who called the curve cyclocylindrique; *Veterum geometria promota in septem de cycloide libras* (Toulouse, 1660). The curve is the intersection of a sphere with a cylinder.

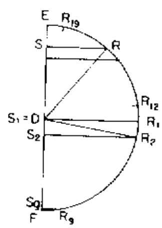
The proof is based on dividing the circumference FME into an indefinite number of equal parts. Then the text continues with the integration of the squares of sines and cosines, as follows.

Proposition 32. *The sum of the squares of the sines on a semicircle is equal to one-eighth of the square of the diameter taken as many times as there are sines.*

Proof. Let the circumference FR_1E (center $D = S_1$) be divided into an infinite number of equal parts, and let all the lines RS represent the sines of the successive arcs [Fig. 4, where the arc R_1F is divided into $n = 10$ equal sections by $R_1R_2 \cdots R_9$, and similarly for the arc R_1E]. Now

$$DR_1^2 = R_1S_1^2 + DS_1^2,$$
$$DR_2^2 = R_2S_2^2 + DS_2^2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Fig. 4



But

$$R_1S_1 = \sin(ER_1)$$
$$R_2S_2 = \sin(ER_2)$$

and

$$DS_2 = \sin(90^\circ - ER_2)$$
$$= R_9S_9.$$

Likewise,

$$R_3S_3 = \sin(ER_3)$$

and

$$DS_3 = R_8S_8 .$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Thus each line DS is equal to a corresponding line RS which is the sine of one arc. Therefore, adding,

$$n \cdot DR^2 = 2 \sum RS^2 \text{ (from zero to } DR_1) = n \cdot \tfrac{1}{4} \cdot EF^2,$$

where n represents the measure of the quadrant arc. But in the semicircle there will be twice as many sines as we have here. Hence

$$\frac{2\cdot\sum RS^2}{2\cdot n\cdot EF^2}=\frac{1}{8},$$

or, since $2n$ is the measure of the semicircle, the sum of the squares of the sines in a semicircle is one-eighth of the square of the diameter multiplied by the number of units in the semicircle.[6]

Proposition 33. *The sum of the squares of the versed sines in a semicircle is three-eighths of the square of the diameter taken as many times as there are versed sines.*[7]

$$\begin{aligned} \textit{Proof.}\ FE^2 &= FS_9^2 + S_9E^2 + 2FS_9\cdot S_9E \\ &= FS_9^2 + S_9E^2 + 2R_9S_9^2, \\ FE^2 &= FS_8^2 + S_8E^2 + 2R_8S_8^2, \end{aligned}$$

Adding:

$$2n\cdot FE^2=\sum FS^2+\sum SE^2+2\Big(2\sum{}' RS\Big),$$

where $2n$ = semicircle FRE; the sum $\sum$ is taken from 0 to FE and $\sum'$ from 0 to RD. Now by Proposition 32, $n\cdot FE^2 = 8\sum' RS^2$; therefore

$$8\Big(2\sum{}' RS^2\Big)=2\sum FS^2+2\cdot\Big(2\sum{}' RS^2\Big),$$

whence

$$6\Big(2\sum{}' RS^2\Big)=2\sum FS^2,$$

or

$$2\sum{}' RS^2=\tfrac{1}{3}\sum FS^2.$$

But by Proposition 32,

$$2\sum{}' RS^2=\tfrac{1}{8}(2n\cdot EF^2).$$

[6] Hence $\int_0^\pi \sin^2\varphi\, d\varphi = \frac{1}{8}\cdot 4\cdot\pi = \pi/2$. Compare this result with that of Pascal (Selection III.7). In those days sines were taken as line segments, whose length depended on the radius R of the circle. The custom of taking $R = 1$, and hence of regarding sines (and cosines, tangents, etc.) as ratios, begins with Euler, *Introductio in analysin infinitorum* (Lausanne, 1748).

[7] The versed sines were introduced by versed $\sin\alpha = R - \cos\alpha$. The companion of the cycloid is a versed-sine curve with respect to BA and BC as axes.

Therefore

$$\tfrac{1}{8}(2n\cdot EF^2) = \tfrac{1}{3}\sum FS^2$$

or

$$\sum FS^2 = \tfrac{3}{8}(2n\cdot EF^2).$$

or, in words, the sum of the squares of the versed sines in a semicircle is three-eighths of the square of the diameter multiplied by the number of units in the length of the semicircle.[8]

Proposition 34. *The volume of the solid generated by the cycloid as it revolves about its base line as an axis is equal to five-eighths of the volume of the circumscribed cylinder.*

In Fig. 2 the lines MM_1, $NN_1, \ldots$, are versed sines, hence it follows from Proposition 33 that

the solid generated by $AN_1DC = \frac{3}{8}$ the cylinder $ABDC$,

but

the solid generated by $AN_1DN_2 = \frac{1}{4}$,, ,, $ABDC$,

and therefore, by addition,

the solid generated by $AN_2DC = \frac{5}{8}$,, ,, $ABDC$.

But AN_2DC is only one-half of the cycloid, therefore the solid generated by the whole cycloid is five-eighths of the whole circumscribed cylinder.[9]

Notice that the solid generated by AN_2DN_1 is equal to the solid generated by the semicircle DC, because these two plane figures have their corresponding lines equal each to each and at the same distance from the axis AC; and the semicircle $DC = \frac{1}{4}$ of the parallelogram $ABDC$, hence the solid $AN_2DN_1 = \frac{1}{4}$ of the cylinder $ABDC$.

11 PASCAL. THE INTEGRATION OF SINES

Roberval was one of the men who influenced Blaise Pascal (on Desargues's influence see Selection III.7), who in his turn wrote a treatise on the cycloid, which he called a roulette (1658). The following paper, which still uses to a certain extent the notion of indivisibles, shows how Pascal integrated $\sin^n \varphi$, $n = 1, 2, 3, 4, \ldots$, making use thereby of a "characteristic triangle," though not yet that of (dx, dy, ds) which we often use now. It is entitled

[8] $\int_0^\pi (1 - \cos\varphi)^2\, d\varphi = \frac{3}{8}\cdot 4\cdot\pi = \frac{3\pi}{2}$.

[9] $\pi\int_0^\pi (1 - \cos\varphi)^3\, d\varphi = \frac{5}{8}\cdot 4\pi\cdot\pi = \frac{5\pi^2}{2}$.

Traité des sinus du quart de cercle (1659); *Oeuvres*, ed. L. Brunschwicg and P. Boutroux (Hachette, Paris, 1914–1921), IX, 60–76.

Pascal's paper differs from that of Roberval: (*a*) in his partial rejection of indivisibles; (*b*) in his more general choice of the limits of the integration interval, so that here we may see a transition to the indefinite integral:

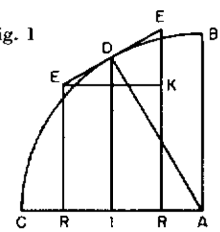
$$\int_{\varphi}^{\pi/2} \sin \varphi \, d\varphi = \cos \varphi, \qquad \int_{\varphi}^{\pi/2} \sin^2 \varphi \, d\varphi = \frac{\pi}{4} - \frac{\varphi}{2} + \frac{1}{2} \sin \varphi \cos \varphi,$$

and so forth.

ON THE SINES OF A QUADRANT OF A CIRCLE

Let ABC [Fig. 1] be a quadrant of a circle of which the radius AB will be considered the axis and the perpendicular radius AC the base; let D be any point on the arc from which the sine DI will be drawn to the radius AC; and let DE be the tangent on which we choose the points E arbitrarily, and from these points we draw the perpendiculars ER to the radius AC.[1]

Fig. 1



I say that the rectangle formed by the sine[2] DI and the tangent EE is equal to the rectangle formed by a portion of the base (enclosed between the parallels) and the radius AB.

For the radius AD is to the sine DI as EE is to RR, or to EK, which is clear because of the similarity of the right-angled triangles DIA, EKE, the angle EEK or EDI being equal to the angle DAI.

Proposition I. The sum of the sines of any arc of a quadrant is equal to the portion of the base between the extreme sines, multiplied by the radius.[3]

[1] The triangle EEK of this figure led Leibniz to his early researches into the calculus; it gave him the idea of the "characteristic" triangle, when EE is small. The segment EE is a tangent in Pascal's essay. With Leibniz it became a chord. For the different forms of this triangle see D. Mahnke, "Neue Einblicke in die Entdeckungsgeschichte der höheren Analysis," *Abhandlungen der preussischen Akademie der Wissenschaften, Kl. Math. Phys. 1* (1925), 1–64.

[2] As with all authors up to the eighteenth century, Pascal's sine of an angle φ is a line, and not a ratio. It is what we now write $R \sin \varphi$, R being the radius of the circle.

[3] This is equivalent to our formula $\int_{\varphi_0}^{\varphi_1} \sin \varphi \, d\varphi = \cos \varphi_0 - \cos \varphi_1$.