Since there are no common terms, let us take out those in which e occurs and let us equate the others:

$$2b^3 - 2b^2a = b^2a, \quad \text{hence} \quad 3a = 2b.$$

Consequently

$$\frac{IA}{AO} = \frac{3}{2}, \quad \text{and} \quad \frac{AO}{OI} = \frac{2}{1},$$

and this was to be proved.[10]

The same method applies to the centers of gravity of all the parabolas ad infinitum as well as those of paraboloids of revolution. I do not have time to indicate, for example, how to look for the center of gravity in our paraboloid obtained by revolution about the ordinate;[11] it will be sufficient to say that, in this conoid, the center of gravity divides the axis into two segments in the ratio 11/5.

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9 TORRICELLI. VOLUME OF AN INFINITE SOLID

Evangelista Torricelli (1608–1647) succeeded Galilei at Florence as mathematician to the grand duke of Tuscany. He was well acquainted with the works of Archimedes, Galilei, and Cavalieri, and corresponded with Mersenne, Roberval, and other mathematicians. He computed many areas, volumes, and tangents, discussed the cycloid, performed what we now see as partial integration, and had an idea of the inverse character of tangent and area problems. He was aware of the logical difficulties in the method of indivisibles (see Selection IV.6). Torricelli is best known as a physicist (we speak of the "vacuum of Torricelli" in the mercury barometer), but his *Opere* (ed. G. Loria and G. Vassura, 3 vols.; Montanari, Faenza, 1919) show his ingenuity also in mathematics. From the *Opere* his manuscript "De infinitis spiralibus" (c. 1646) has been republished (with improved text) with an Italian translation by E. Carruccio (Domus Galilaeana, Pisa, 1955). Our selection is from *De solido hyperbolico acuto* (c. 1643), not published until 1919 in the *Opere*, vol. I, part 1, pp. 191–221. Here we see how he integrated, by a purely geometric method, an integral with an infinite range of integration, but yet finite, something quite remarkable in those days. The method used is that of indivisibles, in this case formed by circles in parallel planes.

ON THE ACUTE HYPERBOLIC SOLID

Consider a hyperbola of which the asymptotes AB, AC enclose a right angle [Fig. 1]. If we rotate this figure about the axis AB, we create what we shall call

[10] These relations were known to Archimedes (see note 8). But Fermat solved this problem on centers of gravity, hence a problem in the integral calculus, with what we might call an application of the principle of virtual variations.

[11] Here ACI of Fig. 3 is rotated about CI.

an acute hyperbolic solid, which is infinitely long in the direction of B. Yet this solid is finite. It is clear that there are contained within this acute solid rectangles through the axis AB, such as $DEFG$. I claim that such a rectangle is equal to the square of the semiaxis of the hyperbola.[1]

We draw from A, the center of the hyperbola, the semiaxis AH, which bisects the angle BAC. This gives us the rectangle $AIHC$, which is certainly a square (it is a rectangle and the angle at A is bisected by the axis AH). Therefore the square of AH is twice the square $AIHC$, or twice the rectangle AF, and therefore equal to the rectangle $DEFG$, as claimed.[2]

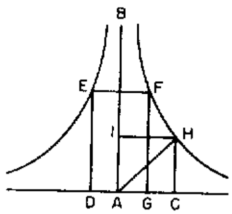
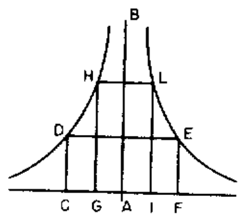
Fig. 1


Fig. 2


Lemma 2. All cylinders described within the acute hyperbolic solid and constructed about the common axis are isoperimetric (I always mean without their bases). Consider the acute solid with axis AB [Fig. 2] and visualize within it the arbitrary cylinders $CDEF$, $GHLI$, drawn about the common axis AB. The rectangles through the axes CE, GL are equal and so the curved surfaces of the cylinders will be equal. Q.E.D.[3]

Lemma 3. All isoperimetric cylinders (for instance, those that are drawn within the acute hyperbolic solid) are to each other as the diameters of their bases. Indeed, in Fig. 2, the rectangles AE, AL are equal, hence $FE : IL = AI : AF$. The cylinder CE has to cylinder GL a ratio composed of $AF^2 : AI^2$ and of $FE : IL$, or of $FA : IA$, or of $FA^2 : AI$ times AF. The cylinders CE, GL are therefore to each other as FA^2 is to AI times AF, and thus as line FA is to line AI. Q.E.D.[4]

Lemma 4. Let ABC [Fig. 3] be an acute body with axis DB, D the center of the hyperbola (where the asymptotes meet), and DF the axis of the hyperbola.

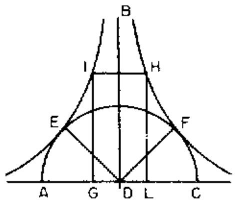
[1] Torricelli speaks of the *latus versum* where we speak of the real axis. The term *latus versum*, or *latus transversum*, is a translation of a Greek term used by Apollonius; see T. L. Heath, *Manual of Greek mathematics* (Clarendon Press, Oxford, 1931), 359. In the present case, taking the rectangular asymptotes as X- and Y-axes (AB the axis of positive Y), the equation of the hyperbola is $xy = \frac{1}{2}a^2$, if the length of the *latus versum* is $2a$.

[2] The theorem used is $xy =$ const., which Torricelli takes (as he remarks in the margin) from Apollonius' *Conics*, II, Prop. 12; see T. L. Heath, *Apollonius of Perga* (Cambridge University Press, Cambridge, England, 1896).

[3] Here Torricelli quotes Archimedes, *On the square and cylinder*, I, Prop. 6; see T. L. Heath, *The works of Archimedes* (Cambridge University Press, Cambridge, England, 1897; reprint Dover, New York).

[4] This reasoning seems rather clumsy to us, since we see immediately that $x_1^2y_1 : x_2^2y_2 = x_1 : x_2$, when $x_1y_1 = x_2y_2$ (= const.). However, to restate this reasoning in the geometric form usual in the seventeenth century (comparing and transforming parallelepipeds) would take as much space as Torricelli needs. The phrase "composed of $AF^2 : AI^2$ and of $FE : IL$" means $(AF^2 : AI^2) \times (FE : IL)$. The text has $IA : AF$, which should be $AF : IA$.

Fig. 3



We construct the sphere $AEFC$ with center D and radius DF. This is the largest sphere with center D that can be described in the acute body. We take an arbitrary cylinder contained in the acute body, say $GIHL$. I claim that the surface of cylinder GH is one-fourth that of the sphere $AEFC$.

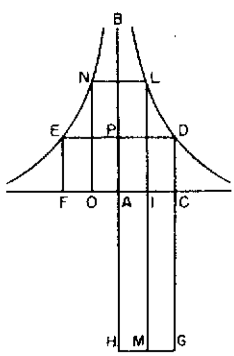
Indeed, since the rectangle GH through the axis of the cylinder is equal to DF^2, hence to the circle $AEFC$, therefore this cylindrical surface $GIHL = \frac{1}{4}$ the surface of the sphere $AEFC$, of which the great circle $AEFG$ is also one-fourth.

Lemma 5. The surface of any cylinder $GHIL$ described in the acute solid (the surface without bases) is equal to the circle of radius DF, which is the semiaxis, or half the latus versum of the hyperbola, for this is proved in the demonstration of the preceding lemma.

Theorem. An acute hyperbolic solid, infinitely long, cut by a plane [perpendicular] to the axis, together with the cylinder of the same base, is equal to that right cylinder of which the base is the latus versum (that is, the axis) of the hyperbola, and of which the altitude is equal to the radius of the basis of this acute body.

Consider a hyperbola of which the asymptotes AB, AC [Fig. 4] enclose a right angle. We draw from an arbitrary point D of the hyperbola a line DC parallel to AB, and DP parallel to AC. Then the whole figure is rotated about AB as axis, so that the acute hyperbolic solid EBD is formed together with a cylinder $FEDC$ with the same base. We extend BA to H, so that AH is equal to the entire axis, that is, the latus versum of the hyperbola. And on the diameter AH we imagine a circle [in the plane] constructed perpendicular to the asymptote AC, and over the base AH we conceive a right cylinder $ACGH$ of altitude

Fig. 4



AC, which is the radius of the base of the acute solid. I claim that the whole body $FEBDC$, though long without end, yet is equal to the cylinder $ACGH$.

We select on the line AC an arbitrary point I and we form the cylindrical surface $ONLI$ inscribed in the acute solid about the axis AB, and likewise the circle IM on the cylinder $ACGH$ parallel to the base AH. Then we have, according to our lemma: (cylindrical surface $ONLI$) is to (circle IM) as (rectangle OL through the axis) is to (square of the radius of circle OM), hence as (rectangle OL) is to (square of the semiaxis of the hyperbola).

And this will always be true no matter where we take point I. Hence all cylindrical surfaces together, that is, the acute solid EBD itself, plus the cylinder of the base $FEDC$, will be equal to all the circles together, that is, to the cylinder $ACGH$. Q.E.D.[5]

Scholium. It might seem incredible that, though this body has an infinite length, yet none of those cylindrical surfaces which we have considered has infinite length. Each of them is limited, as is obvious to anybody who is even moderately familiar with the theory of conics.

The truth of the preceding theorem is sufficiently clear in itself, and it is, I think, sufficiently confirmed by the examples at the beginning of this paper. However, I shall, in order to satisfy in this also the reader who has his doubts about the indivisibles, repeat the same demonstration at the end of this work, in the accustomed way of demonstration as used by the ancient geometers, a way longer, but to me therefore not necessarily safer.[6]

But before we do this, first something else. Since we have given demonstrations about that acute solid of which the asymptotes of the generating hyperbola form a right angle, we shall here in passing state without demonstration to which figures the acute solids are equal, when the asymptotes are at an obtuse, or an acute, angle.

We omit the proofs to avoid ballast; the industrious reader will be able to supply them with little effort.

Let a hyperbola be given of which the asymptotes AB, AC form an obtuse angle. Revolve the figure around the axis AB. Then we will obtain an acute solid, infinitely long toward B, which we cut by a plane DE perpendicular to the axis. Then [Fig. 5] the acute body DBE is equal to the cylinder $DILE$ plus the cone IAL.[7] In Fig. 6, where the intersecting plane is DE, the whole acute solid that stands on the circle DE minus the cone OAV is equal to the cylinder

[5] The reasoning amounts to the evaluation of $\int_0^c 2\pi xy\,dx = \pi a^2 \int_0^c dx = \pi a^2 c$, where $OC = c$. The fact that astonished Torricelli, that the infinite extent of the solid does not imply infinite volume, can be expressed in our language by saying that $\int^\infty dy/y^2$ converges.

[6] The second part of the paper is entitled "On the dimension of the acute hyperbolic solid according to the methods of the ancients." Torricelli says that because of the infinite extent of the solid it is impossible to comprehend it between inscribed and circumscribed solids. Yet he had been able to find a proof with Archimedian methods, and, so he said, had Roberval. The proof takes much more space than that with indivisibles.

[7] If we take AB as the positive Y-axis and the X-axis perpendicular to AB at A, then the equation of the hyperbola is $mx^2 - xy + a^2 = 0$, where $y = mx$ is the equation of the asymptote AL. Then, if $m = -p$, and $DE = 2c$, solid $DBE = 2\pi \int_0^c x(y - y_0)\,dx$, where $-pc^2 - cy_0 + a^2 = 0$, or solid $DBE = \pi(a^2c + \frac{1}{3}pc^3)$. The other theorems of Torricelli can be verified in a similar way.

IE taken together with the cone IAC. Let us now assume that the asymptotes meet at an acute angle, and let plane CD be as in Fig. 7. Then the acute solid CHD together with the cone EAI will be equal to the cylinder $CEDI$. And in Fig. 8 the whole acute solid consisting of the rotation of the mixed infinite quadrangle $ABCDA$ will be twice the cylinder $IEDC$.

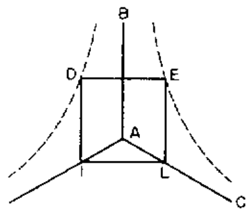


Fig. 5

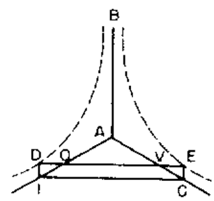


Fig. 6

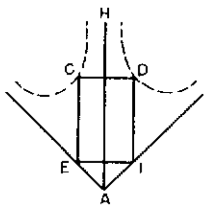


Fig. 7

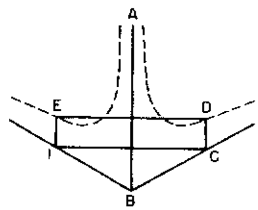


Fig. 8

Then follow 29 corollaries, dealing with special properties of these figures.

Torricelli's paper contains discoveries that he announced to several mathematicians in letters written during 1643. The result was that before the end of the year Roberval, Fermat, and Mersenne were acquainted with them; see E. Bortolotti, *Archivio di storia della scienze 5* (1924), 212–213. Torricelli also discovered ("De infinitis spiralibus") that the arc length of a logarithmic spiral remains finite when it winds an infinite number of times around its asymptotic point; this paper is also reprinted in E. Torricelli, *Opere* (1919). See also on Torricelli's work A. Agostini, "Il problema inverso delle tangenti nelle opere di Torricelli," *Archeion 12* (1930), 33–37, E. Bortolotti, "Le 'Coniche' di Apollonio e il problema inverso delle tangenti de Torricelli," *ibid.*, 267–271, and Selection IV.6, note 6.

The existence of finite areas of infinite extent was already known to scholastic writers and can be found in the so-called sophismata literature of the fourteenth century (Suisseth, Oresme). The subject of this literature was the discussion of logical, mathematical, or physical antinomies (contradictions), which easily involved questions concerning the infinite and the infinitesimal; see A. Maier, *An der Grenze von Scholastik und Naturwissenschaften* (2nd ed.; Edizioni di Storia e Letteratura, Rome, 1953), 264–269, 336–338. An example of such a surface is a step figure (Fig. 9), of which the first step consists of a unit square ABB_1A_1, the second step of a rectangle $B_1C_1C_2B_2$, where C_1 is the center of A_1B_1 and $B_1B_2 = BB_1$, the third step of a rectangle $B_2D_2D_3B_3$, where D_2 is the center of B_2C_2 and $B_2B_3 = B_1B_2 = BB_1$, and so on. The total area is $(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots) = 2$, since all rectangles above A_1B_1 can be placed inside the unit square and "exhaust" it.

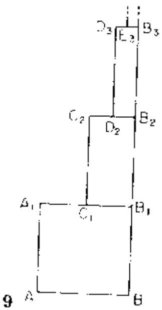


Fig. 9

10 ROBERVAL. THE CYCLOID

The cycloid has been traced back to the French theologian and mathematician Charles Bouvelles (*c.* 1470–*c.* 1553). Galilei was attracted by it, and wrote in a letter to Cavalieri of February 24, 1640 (*Opere*, Edizione nazionale (Barbera, Florence, 1890–1909), XVIII, 153–154): "More than 50 years ago this curved line came to my mind and I wanted to describe it, admiring it because of its most gracious curvature, adaptable to the arches of a bridge. I made several tentative calculations on it and on the space comprised between it and its chord, in order to demonstrate some property. And it seemed at first that such space may be three times the circle which it describes, but it was not that." Galilei gave the curve its name.

About 1630 Father Marin Mersenne (1588–1648), a correspondent of Descartes, Fermat, and many other mathematicians, suggested the cycloid as a test curve for the different methods of dealing with infinitesimals. It soon became one of the most discussed curves of the period, the discussion occasionally leading to acrimonious remarks, so that the curve has been compared to an apple of discord or called the Helen of the geometers. Among those who took up the challenge of Mersenne was Gilles Personne de Roberval (1602–1675), a professor of mathematics in Paris at the Collège du Roi (now Collège de France). From his *Traité des indivisibles* (1634; first published Paris, 1693; reprinted Paris, 1730; Amsterdam, 1736) we present here a section on the cycloid, translated rather freely (the original is somewhat prolix) by E. Walker in *A study of the Traité des Indivisibles* (Teachers College, New York, 1932). It shows how Roberval handled indivisibles,[1] and how he introduced the so-called companion of the cycloid, that is, the sine curve, which was long known under this name, even in the days of Euler. Roberval usually called the cycloid a *roulette*, a custom followed by Pascal; another name was *trochoid* (after Greek *trochos*, wheel). We have, with Walker, used the now customary term cycloid. The interest in this curve was also connected with the age-old speculation concerning the *rota Aristotelis* (see Selection IV.3).

On Roberval see further L. Auger, *Un savant méconnu, G. P. de Roberval* (Blanchard, Paris, 1962). On his mathematics see also C. B. Boyer, *The history of the calculus* (Dover,

[1] It is clear that Roberval, like Cavalieri, uses the method of indivisibles, of which he may have been an independent discoverer (Walker, *A study of the Traité des indivisibles*, 15, 142), but his view was somewhat different. He made clear in his *Traité* that the phrase "the infinite number of points" stands for the infinity of little lines which make up the whole line; see Boyer, *History of the calculus*, 141–142.