Gödel explains what exactly are the three types of basic functions which can be used to define the concept of recursive function: the successor function S, the zero constant function C, and the identity functions U_j^n. These are defined, respectively, by:

$$\begin{aligned} &S(x) = x + 1 \\ &C(x) = 0 \\ &U_j^n(x_1, \ldots, x_n) = x_j. \end{aligned}$$

With this notation, we have, for addition:

$$\begin{aligned} &+(0, y) &&= U_1^1(y) \\ &+(x+1, y) &&= S(U_2^3(x, +(x, y), y)) \end{aligned}$$

and for multiplication:

$$\begin{aligned} &\times(0, y) &&= C(y) \\ &\times(x+1, y) &&= +(U_2^3(x, \times(x, y), y), U_3^3(x, \times(x, y), y)) \end{aligned}$$

A recursive function is a function that can be defined from the basic functions using a finite number of operations of composition and recursion.

Gödel indicates that the concept of general recursive function, which he defines precisely later in the article, is in fact much broader than the earlier idea. The Ackermann function is an interesting example of a general recursive function. This can be defined by the following equations:

$$\begin{aligned} &A(0, y) &&= 1 &&(1) \\ &A(1, 0) &&= 2 &&(2) \\ &A(x+2, 0) &&= S(S(x)) &&(3) \\ &A(x+1, y+1) &&= A(A(x, y+1), y) &&(4) \end{aligned}$$

The fourth equation defines a double recursion, since the recursion depends on two variables. This function is general recursive without being recursive. In fact, it can be shown that the function $h(x) = A(x, x)$ increases faster than any recursive function defined on one variable [4].

15.3 Alonzo Church and Effective Calculability

The two papers by Gödel, as well as the research undertaken by Kleene, were cited by Church in his paper entitled: "An Unsolvable Problem of Elementary Number Theory", presented to the American Mathematical Society in April 1935. In this paper, Church defines another concept of recursive function, whose equivalence to the general recursive function of Gödel was not immediately apparent, but resulted from further work by Kleene.

A. Church

An Unsolvable Problem of Elementary Number Theory, in *The American Journal of Mathematics*, vol. 58 (1936), 345–363.

1. Introduction

There is a class of problems of elementary number theory which can be stated in the form that it is required to find an effectively calculable function f of n positive integers, such that $f(x_1, x_2, \ldots, x_n) = 2$[1] is a necessary and sufficient condition for the truth of a certain proposition of elementary number theory involving $x_1, x_2, \ldots, x_n$ as free variables.

[...]

The purpose of the present paper is to propose a definition of effective calculability which is thought to correspond satisfactorily to the somewhat vague intuitive notion in terms of which problems of this class are often stated, and to show, by means of an example, that not every problem of this class is solvable.

[...]

4. Recursive functions

We define a class of expressions, which we shall call *elementary expressions*, and which involve, besides parentheses and commas, the symbols 1, S, an infinite set of numerical variables $x, y, z, \ldots$, and, for each positive integer n, an infinite set $f_n, g_n, h_n, \ldots$ of functional variables with subscript n. This definition is by induction as follows. The symbol 1 or any numerical variable, standing alone, is an elementary expression. If A is an elementary expression, then $S(A)$ is an elementary expression. If $A_1, A_2, \ldots, A_n$ are elementary expressions and f_n is any functional variable with subscript n, then $f_n(A_1, A_2, \ldots, A_n)$ is an elementary expression.

The particular elementary expressions 1, $S(1)$, $S(S(1))$, … are called *numerals*. And the positive integers 1, 2, 3, … are said to correspond to the numerals 1, $S(1)$, $S(S(1))$, ….

An expression of the form $A = B$, where A and B are elementary expressions, is called an *elementary equation*.

The *derived equations* of a set E of elementary equations are defined by induction as follows. The equations of E themselves are derived equations. If $A = B$ is a derived equation containing a numerical variable x, then the result of substituting a particular numeral for all the occurrences of x in $A = B$ is a derived equation. If $A = B$ is a derived equation containing an elementary expression C (as part of either A or B), and if either $C = D$ or $D = C$ is a derived equation, then the result of substituting D for a particular occurrence of C in $A = B$ is a derived equation.

Suppose that no derived equation of a certain finite set E of elementary equations has the form $k = l$ where k and l are different numerals, that the functional variables which occur in E are $f_{n_1}{}^1, f_{n_2}{}^2, \ldots, f_{n_r}{}^r$ with subscripts $n_1, n_2, \ldots, n_r$ respectively, and that, for every value of i from 1 to r inclusive, and for every set of numerals $k_1{}^i, k_2{}^i, \ldots, k_{n_i}{}^i$, there exists a unique numeral k^i such that $f_{n_i}{}^i(k_1{}^i, k_2{}^i, \ldots, k_{n_i}{}^i) = k^i$ is a derived equation of E. And let $F^1, F^2, \ldots, F^r$ be the functions of positive integers defined by the condition that, in all cases, $F^i(m_1{}^i, m_2{}^i, \ldots, m_{n_i}{}^i)$ shall be equal to m^i, where $m_1{}^i, m_2{}^i, \ldots, m_{n_i}{}^i$ and m^i are the positive integers which correspond to the numerals $k_1{}^i, k_2{}^i, \ldots, k_{n_i}{}^i$ and k^i respectively. Then the set of equations E is said to *define*, or to be a set of *recursion equations* for, any one of the functions F^i, and the functional variable $f_{n_i}{}^i$ is said to *denote* the function F^i.

A function of positive integers for which a set of recursion equations can be given is sa to be *recursive*[2].

It is clear that for any recursive function of positive integers there exists an algorithm u ing which any required particular value of the function can be effectively calculated. For th derived equations of the set of recursion equations E are effectively enumerable, and the algorithm for the calculation of particular values of a function F^i, denoted by a functional variable $f_{n_i}{}^i$, consists in carrying out the enumeration of the derived equations of E until the required particular equation of the form $f_{n_i}{}^i(k_1{}^i, k_2{}^i, \dots, k_{n_i}{}^i) = k^i$ is found[3].

[...]

7. The notion of effective calculability

We now define the notion, already discussed, of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers[4] [...]. This definition is thought to be justified by the considerations which follow, so far as positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion.

It has already been pointed out that, for every function of positive integers which is effectively calculable in the sense just defined, there exists an algorithm for the calculation of its values.

Conversely it is true, under the same definition of effective calculability, that every function, an algorithm for the calculation of the values of which exists, is effectively calculable. For example, in the case of a function F of one positive integer, an algorithm consists in a method by which, given any positive integer n, a sequence of expressions (in some notation) $E_{n1}, E_{n2}, \dots, E_{nr_n}$, can be obtained; where E_{n1} is effectively calculable when n is given; where E_{ni} is effectively calculable when n and the expressions E_{nj}, $j < i$, are given; and where, when n and all the expressions E_{ni} up to and including E_{nr_n} are given, the fact that the algorithm has terminated becomes effectively known and the value of $F(n)$ is effectively calculable. [...]

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[1] The selection of the particular positive integer 2 instead of some other is, of course, accidental and non-essential.

[2] This definition is closely related to, and was suggested by, a definition of recursive functions which was proposed by Kurt Gödel, in lectures at Princeton, N. J., 1934, and credited by him in part to an unpublished suggestion of Jacques Herbrand. The principal features in which the present definition of recursiveness differs from Gödel's are due to S. C. Kleene.
In a forthcoming paper by Kleene to be entitled, "General recursive functions of natural numbers," (abstract in *Bulletin of the American Mathematical Society*, vol. 41), several definitions of recursiveness will be discussed and equivalences among them obtained. In particular, it follows readily from Kleene's results in that paper that every function recursive in the present sense is also recursive in the sense of Gödel (1934) and conversely.

[3] The reader may object that this algorithm cannot be held to provide an effective calculation of the required particular value of F^i unless the proof is constructive that the required equation $f_{n_i}{}^i(k_1{}^i, k_2{}^i, \dots, k_{n_i}{}^i) = k^i$ will ultimately be found. But if so this merely means that he should take the existential quantifier which appears in our definition of a set of recursion equations in a constructive sense. What the criterion of constructiveness shall be is left to the reader. [...]

[4] The question of the relationship between effective calculability and recursiveness (which it is here proposed to answer by identifying the two notions) was raised by Gödel in conversation with the author. [...]

The concept of recursive function defined by Church corresponds to the intuitive idea that an algorithm that provides for calculating a function can be given by a set of equations of a certain type. Consider, for example, the Fibonacci function. The sequence of Fibonacci numbers 1, 2, 3, 5, 8, 13, 21, ... is such that each term is the sum of the two preceding ones. The Fibonacci function which gives the xth term of this sequence is defined by the following set of elementary equations:

$$\begin{aligned} &\mathrm{FIB}(0) = 1 \\ &\mathrm{FIB}(1) = 2 \\ &\mathrm{FIB}(S(S(x))) = +\,(\mathrm{FIB}(x), \mathrm{FIB}(S(x))). \end{aligned}$$

This function can also be defined as a recursive function in the Gödel sense, but this requires advanced techniques.

Church shows how we can effectively calculate the values of a given function from a set of elementary equations. All that is needed is to compute the derived equations obtained by substituting a particular value for a variable (substitution) or by substituting one term of an equation by another term (replacement). Thus, his concept certainly corresponds to the idea of a function whose values are calculable by an effective procedure, that is by an algorithm.

Consider, for example, the set of equations defining the Ackermann function:

$$A(0, y) = 1 \tag{1}$$
$$A(1, 0) = 2 \tag{2}$$
$$A(x + 2, 0) = S(S(x)) \tag{3}$$
$$A(x + 1, y + 1) = A(A(x, y + 1), y) \tag{4}$$

From the derived equations we are able to effectively calculate the values of A. We have:

$A(2, 0) = 4$, etc., $A(x, 0) = 2x$ from (3) by substitution
$A(0, 1) = 1$ from (1) by substitution
$A(1, 1) = A(A(0, 1), 0) = A(1, 0) = 2$ from (4) by replacement
$A(2, 1) = A(A(1, 1), 0) = A(2, 0) = 4 = 2^2$ from (4) by replacement
$A(3, 1) = A(A(2, 1), 0) = A(4, 0) = 8 = 2^3$ from (4) by replacement
$A(x, 1) = A(A(x - 1, 1), 0) = A(2^{x-1}, 0) = 2\times 2^{2x-1} = 2^{2x}$ from (4)
$A(0, 2) = 1$ from (1) by substitution
$A(1, 2) = A(A(0, 2), 1) = A(1, 1) = 2$ from (4) by replacement
$A(2, 2) = A(A(1, 2), 1) = A(2, 1) = 2^2$ from (4) by replacement
$A(3, 2) = A(A(2, 2), 1) = A(2^2, 1) = 2^{2^2}$ from (4) by replacement
$A(x, 2) = A(A(x - 1, 2), 1) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ from (4) by replacement
etc.

In his paper, Church explicitly poses the question as to whether the concept of recursiveness corresponds to the notion of being effectively calculable. He indicates, in a note, that this question was raised in a conversation that he had had with Gödel. The latter had already pointed out, in his 1934 paper, that every recursive function, in his sense, is computable by a finite procedure. We also find, in a footnote to the paper, doubtless added later, that "the converse appears to be true", provided that the concept of general recursive function is understood in the sense that it is used in the paper.