E9 – De linea brevissima in superficie quacunque duo quaelibet puncta iungente

CASP 3 (1728) 1732, 110-124
OO I.25, 1-12
On the shortest curve on a surface that joins any two given points

Even Euler had to do homework. Sometime in 1728, his mentor at the University of Basel, Johann Bernoulli assigned him to find “the shortest line between two given points on a surface.” A copy of the manuscript that Euler handed in to Bernoulli is in the archives in Moscow (though that seems to be a copy, as it is not in Euler’s handwriting). Euler then published a paper based on the homework in the *Commentarii* of 1728.

The manuscript and the paper together lay the first analytical foundations for the calculus of variations. There had been earlier results in the calculus of variations. Some people trace its origins at least to Dido’s solution of the isoperimetric problem as recounted in Virgil’s *Aeneid*, and certainly Johann Bernoulli’s own solution to the brachistochrone problem was a masterful contribution to the subject. There was something *ad hoc* about Bernoulli’s solution, though, so Euler’s work might be the first general analysis and general principles of problems in the calculation of variations. As such, it should perhaps be considered as an alternative to Tom Banchoff’s nomination for “The Best Homework Ever?”

Both the manuscript and the paper are reprinted in the *Opera Omnia*, I.25. The manuscript takes 8 pages and consists of 20 paragraphs. The paper itself is 12 pages, 36 paragraphs. The two are distinct, but quite similar. Euler does not “recycle” any paragraphs from the manuscript, but he keeps the same notation in both places. He works his examples in more detail in the manuscript, and includes more steps in his derivations. The paper is more general and covers a little more material. In what follows, we will not always try to distinguish between them.

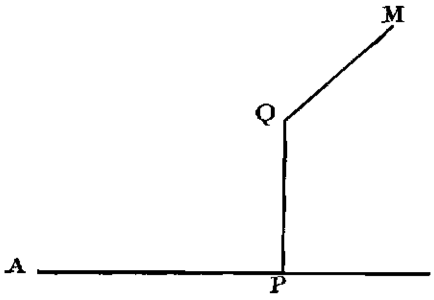
Constantin Caratheodory, in his capacity as editor of the volume of the *Opera Omnia* on calculus of variations writes that “Beginning in paragraph 14, this work reads like a worksheet, and one watches Euler’s discoveries as he makes them.” It is a thrill to watch the Master at work.

Euler begins with a statement of the problem and with the two easy special cases. If the surface is a plane, then the shortest line between two given points is a straight line, and if the surface is a sphere, then the shortest line is a segment of a great circle. Euler uses the word “linea” to mean either a curve or a straight line, and we will follow in his usage by using the word “line” in the same way.

He then notes that for other surfaces “convex or concave or mixed of these”, no general solution is known, and “Johann Bernoulli posed this question to me, of finding a universal means of finding the equation of that shortest line determined on whatever surface might be proposed.”

If the surface happens to be convex, then Euler notes that there is what he calls a “mechanical solution”. He can solve the problem by stretching a string between the two given points. Euler notes that this method gives “chords on the convex parts”, though, and that this method only gives the curve, and not an equation describing the curve. He does not note that the solution thus found may be a local minimum and not a global minimum.

This kind of mechanical solution to a problem seems unusual for Euler. He does not indicate if it is his own idea or if he got it from somebody else, perhaps Bernoulli. In any case, the example indicates that Euler is willing to use any analytical tool to solve a problem and that, to him, anything is fair in mathematics.

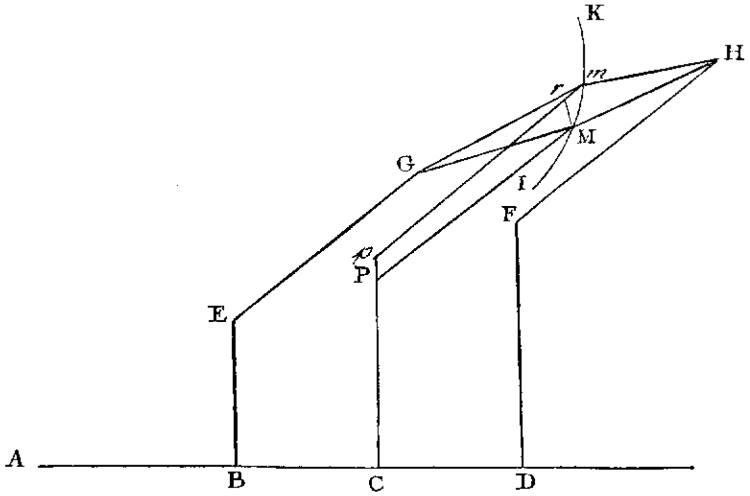


In his solution, Euler is going to use what is then a new system of three-dimensional coordinates, a system that he attributes to Hermann. This sytem is illustrated in Figure 1. He describes the position of a point M relative to a point A. Along a line AP, he finds the point P so that M is in the plane perpendicular to the line AP at the point P. Then, he further finds the point Q so that the line segment PQ is perpendicular to the segment QM. He does not specify what line to choose for AP, nor in what plane the points A, P and Q should lie. Rather, these are left to be chosen for the convenience of the problem. Finally, he denotes t=AP, x=PQ and y=QM.

All these choices seem arbitrary, but he will describe other points in the problem relative to the same line containing A, P and the same plane containing A, P, Q.

Now, since this is a problem in arc length, he lets AM=a and notes that $a^2 = t^2 + x^2 + y^2$.

Now in paragraph 8 of the 32 paragraphs in the article, Euler notes that “an equation will define the surface” and that an interseciton of equations will define a curve on the surface. He will be interested in two kinds of such intersections. He will take sections by holding one of the three variables constant, usually the variable t. This will give him a curve in a plane. He will also give a solution in one case by giving an equation in terms of t and x.



Now, Euler moves to Figure 2. The reader will recognize that the coordinate system introduced in Figure 1 is being used here. The points G, I, M, m, K and H are all on the surface, and the other points, with the exception of r, are part of the coordinate system. The curve IMmK is the curve on the surface determined by its intersection with the plane perpendicular to AC and passing through C. Euler wants to find necessary conditions on the point M that minimize the length GM+MH, and he announces that to do this, he will use the “method of maxima and minima”. To this end, Euler tells us that if M minimizes GM+MH and if m is a point on the curve IMK near M, then GM+MH=Gm+mH.

To a modern eye, this seems exactly wrong. We want to say that if M is the minimum then GM+MH<Gm+mH. Euler, though, is thinking of differentials. The quantity (GM+MH)-(Gm+mH) is a differential of the arc length at the point M, and if M is a minimum, than that differential will be zero. With this interpretation Euler’s equation is correct.

Having declared this plan, Euler assigns variables. BC=CD=a, BE=b, EC=c, DE=F, FH=g, CP=x, PM=y. Then Cp=x+dx and pm=y+dy. Now, Euler calculates.

$$GM = \sqrt{a^2 + (x-b)^2 + (y-c)^2}$$

$$GM^2 = (PM - GE)^2 + (CP - BE)^2 + BC^2$$

and likewise

$$HM = \sqrt{a^2 + (f-x)^2 + (g-y)} \ .$$

Adding

$$GM + MH = \sqrt{a^2 + (x-b)^2 + (y-c)^2} + \sqrt{a^2 + (f-x)^2 + (g-y)^2} \ .$$

Differentiate (actually, take differentials) and set equal to zero to get

$$\frac{(x-b)dx + (y-c)dy}{\sqrt{a^2 + (x-b)^2 + (y-c)^2}} = \frac{(f-x)dx + (g-y)dy}{\sqrt{a^2 + (f-x)^2 + (g-y)^2}} \ .$$

Now, Euler makes a common confusion between necessary conditions and sufficient conditions and says that from this, the point M is determined.

Now we reach section 14, the point at which Carathéodory says that we can see the way Euler thinks.

The curve IK is determined by the surface and the point C, and can be given in terms of the coordinates x and y. So we can differentiate the equation of the curve IK and get some differential equation Pdx=Qdy. Euler also gives this in a second form as ratios: dx:dy=Q:P. Substituting these in the equation above produces

$$\frac{(x-b)Q + (y-c)P}{\sqrt{a^2 + (x-b)^2 + (y-c)^2}} = \frac{(f-x)Q + (g-y)P}{\sqrt{a^2 + (f-x)^2 + (g-y)^2}}$$

which is, as Euler says “without differential quantities”.

Making the repertoire of substitutions BC=CD=a=dt, DF=f=x+dx, FH=g=y+dy. BE=b=a-dx+ddx and EG=c=y-dy+ddy gives

$$\frac{Q(dx-ddx)+P(dy-ddy)}{\sqrt{dt^2+(dx-ddx)^2+(dy-ddy)^2}}=\frac{Qdx+Pdy}{\sqrt{dt^2+dx^2+dy^2}}.$$

But, by the application of the same Method of Maxima and Minima used earlier, this just means that the differential of the quantity on the right, $\frac{Qdx+Pdy}{\sqrt{dt^2+dx^2+dy^2}}$ is zero. So, Euler holds P, Q and dt constant, in order to stay on the sme section curve of the surface, and apply the quotient rule to get, two steps later

(1.1) $$\frac{Qddx+Pddy}{Qdx+Pdy}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}.$$

Moreover, Euler can describe the surface by a differential equation

(1.2) $$Pdx=Qdy+Rdt$$

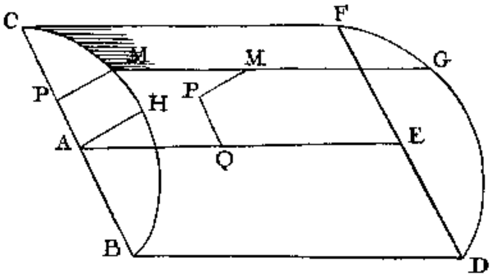
When the surface is constrained to the curve IK, then dt=0 so the equation reduces to Pdx=Qdy.

Now, Euler's analysis for the general case concludes that the shortest line on a surface must satisfy the two differential equations (1.1) and (1.2).

Euler applies his analysis to three kinds of surfaces:

1. Cylinders, not necessarily circular cylinders
2. Circular cones
3. Surfaces of revolution.

Euler illustrates the case of the cylinder with Figure 3. There, he assigns variables AQ=t, QP=x and PM=y. Note that the variables are the same, but the points are labelled slightly differently than they were in Figure 2. Euler explains that the differential equation for this surface is Pdx=Qdy, and does not depend on t. He further notes that if the cylinder is an ordinary circular cylinder in which the curve BHC is a circle with A as its center, then the differential equation has is xdx=-ydy. These relatively simple forms allow Euler to proceed with his analysis.



He substitutes P:Q=dy:dx into equation (1.1) to get

(1.3) $$\frac{dxddx+dyddy}{dx^2+dy^2}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}.$$

Integrating (1.3) gives

$$\sqrt{dx^2+dy^2}=m\sqrt{dt^2+dx^2+dy^2}$$

or

$$dx^2+dy^2=nndt^2$$

[This next section requires a bit more of my attention. I can't quite make it work for me.]

It is not entirely clear, in this calculation, why Euler can assume that m<1, so the quantity $\frac{1}{1-m^2}$, which he replaces with *nn* is positive.

Integrating the square root of this last equation gives $nt = \int \sqrt{dx^2 + dy^2} + C$, from which Euler concludes that "t is always proportional to an arc in a fixed section." Euler recognizes that this may not be perfectly clear, so he clarifies it a bit in paragraph 22.

Euler looks at the integral equation $nt = \int \sqrt{dx^2 + dy^2} + C$ a little more closely and tries to interpret the value of n. In the case n=0, then this is describing the shortest arc connecting two points on the same line of the cylinder parallel to the axis AE. In the case n=1, then this describes an arc connecting two points on the same seciton. Then the shortest curve traces that section.

[End of my confusion]

Euler now specializes to the case of a circular cylinder, so he can substitute xdx=-ydy and xx+yy=aa. then

$$nt + b = \int \sqrt{dx^2 + dy^2} .$$

Differentiating both sides and squaring gives

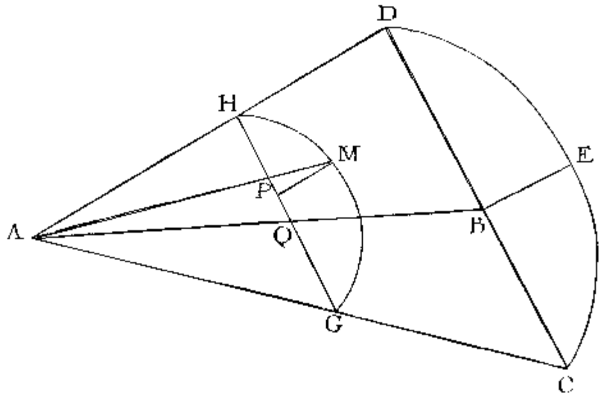
$$n^2 dt^2 = dx^2 + dy^2 .$$

Now, substituting for y gives

$$ndt = \frac{adx}{\sqrt{a^2 - x^2}} \text{ or } nt = \int \frac{adx}{\sqrt{a^2 - x^2}}$$

which Euler recognizes as a sine. Euler concludes that the shortest path on a cylinder traces a piece of a sine curve as its projection onto a plane containing the t-axis.

Euler's next example is a cone, as illustrated in Figure 4. There, AQ=t, QP=x and PM=y. He notes that all transverse sections are similar. That is to say the curve HMG is similar to the curve DEC. Euler further notes that "If in the equation, nt, nx, ny are put in the place of t, x, y, then the equation is not changed. This property is that of homogeneous equations, in which t, x and y all make the same dimension." It is interesting that Euler knew this geometric similarity property of homogeneous equations, and that he uses the term "dimension" in this context rather than "degree".

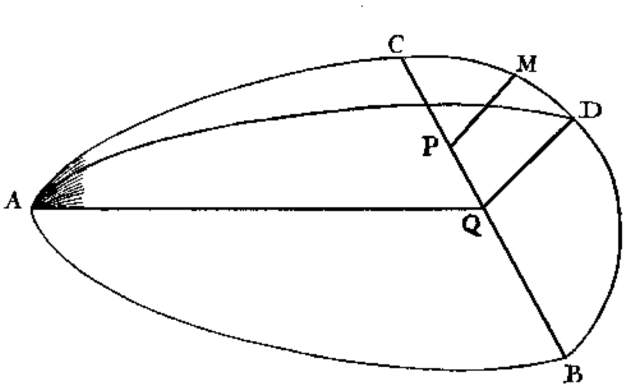


From here, Euler's analysis procees pretty much as before. Because the form of the surface is a cone, he knows the equation of the surface is of the form tF(x,y), and so he is able to make substitutions as before.

Euler's final example is a surface of revolution, as illustrated in Figure 5. He first notes that this example is different in nature from the first two, since both the cylinder and the cone could be flattened out into a plane. The shortest path could then be found on the plane and the solution carried back to the surface itself. The special properties of a surface of revolution give Euler enough tools to work his magic.

Euler assigns variables AQ=t, QP=x and PM=y. Since we have a surface of revolution "if I hold t constant, or dt=0, then it will be the equation of a circle xx+yy=Const. or xdx=-ydy. From this, we see that the equation for a solid of revolution is xx+yy=T, where t denotes some function of that variable t, and of constants." From this, Euler gets that the differential equation that describes the surface will be $xdx = -ydy + Rdt$, "in which R depends only on t and on constants."

We see from Euler's choice of words here that Euler does not yet clearly distinguish between the solid and its surface. Moreover, he still describes the function T as being a function of t *and of constants*.



To continue his calculation, Euler goes back to the fundamental equation (1.1)

$$\frac{Qddx+Pddy}{Qdx+Pdy}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$$

He makes his substitutions to get

$$\frac{xddy-yddx}{xdy-ydx}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$$

He integrates this to get

$$l(xdy-ydx)=l\sqrt{dt^2+dx^2+dy^2}+la$$

where we see that Euler uses the symbol "*l*" to denote what we now call the natural logarithm function. Finally, he exponentiates both sides of this to get

$$(1.4)\qquad xdy-ydx=a\sqrt{dt^2+dx^2+dy^2}$$

and "This, together with the natural equation of the surface expressed as $xdx=-ydy+Rdt$, will determine the shortest line."

Though Euler now has a general solution for the shortest line on a surface of revolution, he still has a few more fruits to be harvested from his efforts. He notes that the value of *a* depends on the particular points on the surface and considers what happens if a=0. In this case, equation (1.4) gives $xdy=ydx$, and so $y=nx$. "And so it is known that the perphery of the curve rotated about the axis represents the shortest line between its endpoints." Euler points out that this implies that the shortest line between two points on a sphere is the arc of a great circle, confirming what he already knew when he started the paper.

Euler's concluding paragraph reads:

> "The Celebrated Johann Bernoulli proposed this question to me and urged me to write up my solution and to investigate these three kinds of surfaces which lead to solutions that are integrable equations. I wanted to include the solutions to these questions because they followed so easily from what I had done earlier."

In other words, he says "Once I knew what I was doing, this homework was pretty easy."

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