DÉTERMINATION

DE LA

DEMI-CIRCONFÉRENCE D'UN CERCLE,

DONT LE DIAMÈTRE EST = 1, EXPRIMÉE EN 140 FIGURES DÉCIMALES.

Par

M. *GÉORGE VEGA.*

Préſenté à l'Académie le 20 Août, 1789.

L'Académie croit pouvoir ſe diſpenſer d'inſérer ici tout le calcul long & pénible par lequel l'auteur eſt parvenu à la valeur de π, ou de la demi-circonférence d'un cercle dont le diamètre eſt = 1; il ſuffit de tranſcrire ici la double ſérie infinie dont il s'eſt ſervi pour cet objet, & qui eſt (*)

Why 140 Digits of Pi Matter

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Abstract

Among mathematicians, Jurij Vega is best known for his calculation of π to 140 decimal places. Although this achievement is only a small part of his four-volume *Vorlesungen über die Mathematik*, and his ten-place logarithmic tables were far more useful, the value of π captured the imaginations of mathematicians. Here, we will look at how and why Vega performed his famous calculation, and what makes it important.

Vega's Announcement

Vega announced his value of π to the world in an article in volume IX of the *Nova Acta Academiae Scientiarum Imperialis Petropolitanae*. He had presented the article to the Academy (that is, it was read before the Academy; Vega did not have to attend) at the weekly meeting of 20 August 1789. The article itself appeared in the volume of works for the year 1791, and but that volume was not actually published until 1795. All this makes it confusing to try to attach a meaningful date to the work. Sitar, for example, reports the date as 1795, while the St. Andrews web site on the history of mathematics, MacTutor, gives it as 1789.

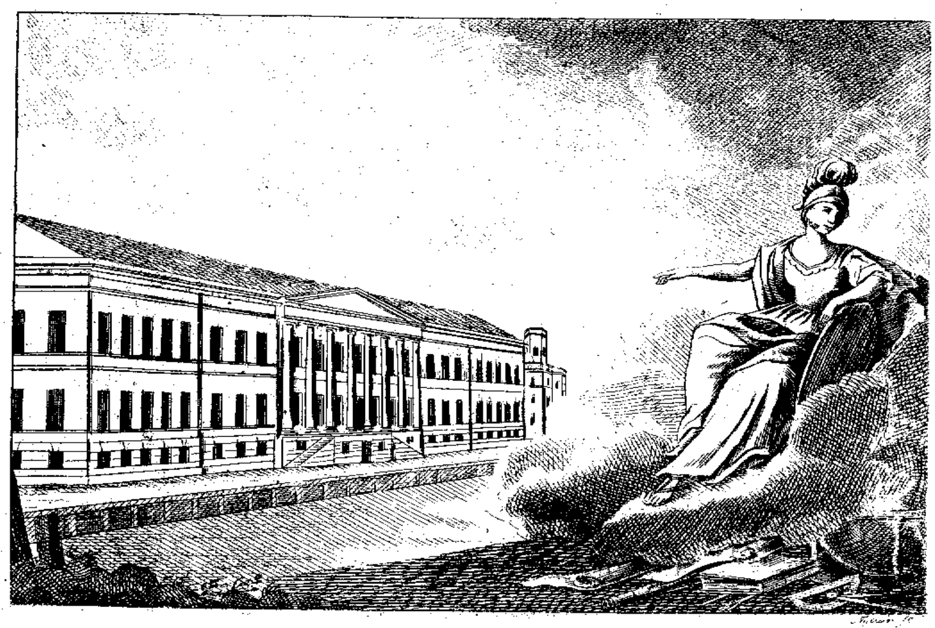
To complicate putting a date on Vega's work on π even further, the article in the *Nova Acta* is based on work that Vega had previously published in the *Vorlesungen* and in his Logarithmic Tables.

Some members of the scientific community may have known of Vega's work before it appeared in the *Nova Acta*. Most, though, would not have known it, since most of his work had been published only in German. This was a time when the Language of Science was shifting from Latin to French. Many who might have been interested would not have been able to read his work in German readily. Moreover, the heart of the work on π is in the middle of the much longer *Vorlesungen*, and so was easy to overlook. So, this article in the *Nova Acta* was Vega's first work addressed to a first-rate international audience.

NOVA ACTA
ACADEMIAE SCIENTIARVM IMPERIALIS PETROPOLITANAE

TOMVS IX.

PRAECEDIT HISTORIA EIVSDEM ACADEMIAE
AD ANNVM MDCCXCI.



PETROPOLI
TYPIS ACADEMIAE SCIENTIARVM MDCCXCV.

The Great Academies

The 18th Century had been a great era for scientific academies. Four societies dominated the scientific scene, the Royal Society in England, the Petersburg Academy in Russia, Frederick II's Berlin Academy, and the Paris Academy. By the 1790's, the age of the Academies was ending.

The Berlin Academy began to diminish with the death of Maupertuis in 1759, and then suffered even more when Euler left Berlin for St. Petersburg in 1766. Though the great Lagrange replaced Euler in Berlin, the Academy continued to lose its energy, and it lost much of its support when Frederick II died in 1786.

The Paris Academy had been swallowed in the turmoil of the Revolution of 1789, but its decline had begun in the 1760's because of a chronic shortage of funds. Beginning in the 1720's, the Paris Academy had steered much research in science and mathematics with its annual Paris Prize. The Prize was funded by an endowment, invested in Paris real estate. The revenues from the real estate were split between the Prize and the funding of the other operations of the Academy. No revenues were designated to maintain the real estate, so by the 1760's, the properties were falling into disrepair and revenues were declining. Beginning in 1762, the Paris Prize was contested only every second year, and that practice continued until the Academy was dissolved in 1793 in the wake of the Revolution.

Meanwhile, the Royal Society in England had spent much of the 18th Century in the shadow of Isaac Newton. England then, as it is now, was an island, and England often took its isolation to go its own way, independent of what the Continent was doing. So it was in scientific affairs. The English generally did not invite foreign members to their Society unless they were already important members of other important societies. Continental societies accepted foreign members on the basis of their accomplishments as well as on the basis of the awards and titles they had accrued. This meant that the offices of the Royal Society were closed to Jurij Vega unless and until he was first recognized by the great Continental academies.

Of the four Great Academies, this leaves the Petersburg Academy. It was not the academy it had been before Euler died in 1783, but it still enjoyed the patronage of Catherine II, until her death in 1796, and the presidency of her favorite courtier, Princess Etkaterina Dashkova. Moreover, Euler had left a legacy of well over 250 articles to fill the pages of the *Nova Acta* for decades after his death. The Academy continued to publish his articles in every volume of their journals until 1820, and the last of them was only published in 1862, 79 years after Euler died.

Thus, it seems that the St. Petersburg Academy was the healthiest of the Great Academies on the continent of Europe, and the Royal Society was closed to continental scientists who were not already recognized and acclaimed elsewhere.

From all this, we should conclude that the pages of the journal of the St. Petersburg Academy was the best possible place for Jurij Vega to publish his results.

The *Nova Acta* of the Petersburg Academy

As mentioned above, Vega's paper on π appeared in the *Nova Acta* of the St. Petersburg Academy. It is worthwhile to look at the journal itself. The *Nova Acta* was the fourth series of journals that the Academy published, beginning with the *Commentarii* in the 1720's. Every 15 or 20 years, the Academy would discontinue a series and begin a new one. This practice continued until the Russian Revolution of 1914, which ended the publication of the Academy's eighth series.

A volume of the *Nova Acta* covered the activities of the Academy for one year, and was usually published between three and ten years later, depending on political and economic circumstances. Volume IX was for the year 1791, and was published in 1795.

Each volume was about 600 pages long. The bulk of the volume, the last 400 pages, were the *Acta* themselves, the scholarly papers written by members of the Academy. This volume includes ten posthumous papers by Euler, two by his son, Johann Albrecht, who ran the observatory in St. Petersburg for many years after his father's death, and thirteen other articles on mathematics, biology and astronomy by eight other members of the Academy.

The first 200 pages, called the *Histoire*, told about the formal activities of the Academy. We find news of the Prize Problem that had been posed in 1788 for the year 1791, on the chemical nature of color. There were only five entries. The Academy judged just two of the papers to be worthy, but decided to continue the problem until 1795, and added some remarks to clarify what they were looking for.

This is followed by a proclamation by Princess Dashkova, President of the Academy, establishing a pension for aged and sick members of the Academy.

Other business of the Academy included the acquisition of books and specimens for the library and museum, deaths of members of the Academy, new members, and accounts of the weekly meetings. Over half of the *Histoire*, 120 out of 200 pages in Volume IX, is taken up by treatises presented to the Academy by "Savans étrangers & approuvés dans les Assemblées."

Vega's Paper in the *Nova Acta*

It is among these treatises by "Savans étrangers" that we find Jurij Vega's "Détermination de la démi-circonférence d'un cercle dont le diamètre est =1, exprimée en 140 figures decimals."

It is a short article, only four pages. In brief, it says that Vega used a "double series:"

$$\pi = 8 \times \left\{ \begin{array}{l} \frac{73}{1.3}a + \frac{169}{5.7}b + \frac{265}{9.11}c + \frac{361}{13.15}d + \frac{457}{17.19}e \\ \qquad + \frac{553}{21.23}f + \frac{649}{25.27}g + \&c. \\ + \frac{26}{1.3}A + \frac{58}{5.7}B + \frac{90}{9.11}C + \frac{122}{13.15}D + \frac{154}{17.19}E \\ \qquad + \frac{186}{21.23}F + \frac{218}{25.27}G + \&c. \end{array} \right\}$$

A footnote, inserted by the editors of the *Nova Acta*, tells us that Euler discovered a different series, and that the article in which Euler explains this would be inserted into Volume X of the *Nova Acta*. In fact, it was not inserted until Volume XI, the volume for 1793 that appeared in 1798. Volume X was a memorial volume, observing the death of Catherine II in 1796, and had no scientific content. It was the only volume published by the Petersburg Academy between 1730 and 1820 that contained no papers by Euler. The footnote tells us that Euler's series was

$$p = \left\{ \begin{array}{l} \frac{28}{10}\left[1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{2}{100}\right)^3 + \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9}\left(\frac{2}{100}\right)^4 + \&c.\right] \\ + \frac{30336}{100000}\left[1 + \frac{2}{3}\left(\frac{144}{100000}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{100000}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{144}{100000}\right)^3 + \&c.\right] \end{array} \right\}$$

Vega used a different series because Euler's series converges more slowly. The editor's footnote describes Euler's series as "à la vérité, n'est pas si convergente que celles de M. le Major Vega."

Indeed, both series do converge rapidly. The table below shows the values calculated by a spreadsheet (Excel) for the first several approximations. We see that after just eight iterations, Euler's approximation has reached 14 decimal places, the limit of the precision of the spreadsheet, but Vega's approximation reached the same accuracy after just seven (or six) iterations.

	Euler's series	Vega's series
0	3.10336000000000	
1	3.14098455893333	3.13544253680308
2	3.14158222775856	3.14155123586771
3	3.14159246817265	3.14159230145587
4	3.14159265021762	3.14159265027498
5	3.14159265352752	3.14159265355673
6	3.14159265358863	3.14159265358945
7	3.14159265358977	3.14159265358979
8	3.14159265358979	

Origins of Vega's Ideas

The article in the *Nova Acta*, though, does not tell us *why* these series converge to π. For this, we must follow the footnotes. Vega refers us to Volume II of his *Vorlesungen*, while the editors send us to Euler's article that would appear three years later. We will follow Vega's own footprints first.

In Volume II of the *Vorlesungen*, Vega tries to lift students from algebra to calculus in just one book. At over 700 pages, it is the longest by far of the four volumes of the *Vorlesungen*. (Volume I was 475 pages; Volumes III and

IV were 433 and 320, respectively.) Chapter 4, of seven, is a fast-paced account of trigonometry, from the point of view of functions. A student who reads only Chapter 4 might think what trigonometry has more to do with functions and series than with triangles, but Vega devotes Chapters 5 and 6 to practical mathematics. There, he shows that trigonometry really is about triangles by detailing applications such as surveying and mechanics.

Vega's presentation is thorough and by no means elementary. A mark of his thoroughness is that he includes among his trigonometric functions the Chord. This had been the fundamental trigonometric function for Ptolemy, almost 2000 years earlier. By Vega's time, it was seldom used, and now it has slipped entirely from the curriculum. He also uses the versine, another now-forgotten trigonometric function that only vanished about 50 years ago.

He is writing a text for an engineering style audience, so he does not dwell much on the mathematical formalities of theorems and proofs. About twenty pages into Chapter 4, he begins to describe ways to use series to evaluate trigonometric functions. He had done a little bit of this a hundred pages earlier, where he evaluated a series to find π to 35 decimal places.

Vega reverses the usual presentation of these series, though, by first giving the series for the arcsine of z,

$$z = \arcsin x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot x^5}{2 \cdot 4 \cdot 6 \cdot 7} + \ldots$$

Then, using methods he had given near the end of Volume I for inverting such series, he gets the series for the sine, the point at which most presentations begin:

$$x = \sin z = z - \frac{z^3}{2 \cdot 3} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \ldots$$

.

He uses this to find sin 1' to ten decimal places. Since 1 minute is so close to zero, he gets ten decimal places of accuracy with just two terms of the series.

Except for the order of things, this material is fairly standard. He adds some less familiar material when he describes some results of Euler from a relatively obscure 1739 paper, “Methodus facilis computandi angulorum sinus ac tangents tam naturals quam artificialis”. Both Euler and Vega describe series that give $\sin \frac{m}{n} 90^\circ$ and $\cos \frac{m}{n} 90^\circ$ to eleven decimal places in just eight terms.

A few pages later, Vega turns to series for inverse trigonometric functions. He begins with a series for the arctangent that was known to Leibniz:

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11}...$$

For $x = 1$, this gives $\frac{p}{4}$, but to get even three decimal places of accuracy, one must evaluate 2000 terms. This is a result for a mathematician, not for a practical man like Vega. For small values of *x*, the series converges much more rapidly. Vega, following Euler, sets out to exploit this.

Two pages later, after investing a bit of time in some trigonometric identities, Vega derives

$$\arctan a = \arctan b + \arctan \frac{a-b}{1+ab}\cdot$$

This identity can convert a single series, like the arctan series above, into two arctangent series with smaller arguments, so they will converge faster. Vega first takes $a = 1, \quad b = \frac{1}{3}$, so that $\frac{a-b}{1+ab} = \frac{1}{2}$, to get

$$
\begin{aligned}
\frac{p}{4} &= \arctan 1 \\
&= \arctan\frac{1}{2} + \arctan\frac{1}{3} \\
&= \left(\frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \ldots\right) + \left(\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \ldots\right)
\end{aligned}
$$

This series gives π to 15 decimal places in 21 steps, and what takes the Leibniz series 2000 terms, takes only four terms of each series. In fact, since the first series converges more slowly than the second, one would do slightly better by taking about 50% more terms of the first series than of the second.

Vega tells us that "Diese Reihe wurde von Euler angegeben." He is referring to a 1737 paper by Euler titled "De variis modis circuli quadraturam numeris proxime expremendi." In that paper, Euler developed the techniques that Vega describes, and, for this series, he estimates that 154 terms would give 100 decimal place accuracy. Euler, though, did not do the calculations.

Vega continues, as Euler suggests, by expanding the second part of the double series, taking $a = \frac{1}{2}, \quad b = \frac{1}{3}$, so that $\frac{a-b}{1+ab} = \frac{1}{7}$. This gives $\arctan\frac{1}{2} = \arctan\frac{1}{3} + \arctan\frac{1}{7}$. It was clever to take $b = \frac{1}{3}$, since that makes what would have been a triple series into a double series, as follows:

$$
\begin{aligned}
\frac{p}{4} &= \arctan\frac{1}{2} + \arctan\frac{1}{3} \\
&= 2\arctan\frac{1}{3} + \arctan\frac{1}{7}
\end{aligned}
$$

Since $\frac{1}{3}$ is smaller than $\frac{1}{2}$, the double series arising from this identity converges faster than the first one. This is essentially the series Vega uses. His calculations go as follows:

$$
\begin{aligned}
\frac{p}{4} &= 2\arctan\frac{1}{3} + \arctan\frac{1}{7} \\
&= 2\left(\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \frac{1}{9\cdot 3^9} - \ldots\right) + \left(\frac{1}{7} - \frac{1}{3\cdot 7^3} + \frac{1}{5\cdot 7^5} - \frac{1}{7\cdot 7^7} + \frac{1}{9\cdot 7^9} - \ldots\right)
\end{aligned}
$$

Here, Vega's calculations diverge from Euler's. Vega combines each positive term of each series with the negative term following it, to "telescope" the series. In the first series, for example, the first pair of terms in the first series becomes $\frac{1}{3}-\frac{1}{3\cdot 3^3}=\frac{26}{81}=\frac{26}{1\cdot 3}\cdot\left(\frac{1}{3^3}\right)$. Similarly, the first pair in the second series becomes $\frac{73}{1\cdot 3}\left(\frac{1}{7^3}\right)$. If we continue this telescoping process for both series, and if we multiply both sides of the equation by four, we get exactly the series Vega describes in his paper in the *Nova Acta*.

In the *Vorlesungen*, Vega derives two more particularly interesting identities. He shows that

$$\frac{p}{4}=5\arctan\frac{1}{7}+2\arctan\frac{3}{79}.$$

Since the denominators of these fractions are so large, the corresponding series converge more rapidly than any series we have seen so far. They give 15 decimal place accuracy in just 8 steps.

Vega also finds that

$$\frac{p}{4} = 5\arctan\frac{1}{13} + 5\arctan\frac{1}{21} + 2\arctan\frac{1}{31} + 2\arctan\frac{1}{43} + 3\arctan\frac{1}{57}.$$

The denominators here are even larger, and it would converge about four times faster than the previous double series, but also each step here is more than twice as complicated as the previous series would have been. One wonders why Vega did not use one of these series instead of the one he did use. We cannot know, but perhaps it was because by the time he discovered these series, he had already started, or even completed, the calculation with the series he did use.

D'où l'auteur a calculé la valeur ſuivante pour
π = 3 . 14159 . 26535 . 89793 . 23846 . 26433 .
83279 . 50288 . 41971 . 69399 . 37510 .
58209 . 74944 . 59230 . 78164 . 06286 .
20899 . 86280 . 34825 . 34211 . 70679 .
82148 . 08651 . 32823 . 06647 . 09384 .
44767 . 21386 . 11733 . 138 ..

Why 140 Decimal Places Matter

Now that we have seen the technical details of how Vega calculated π to 140 decimal places, it makes sense to ask why he undertook such a long and difficult calculation. There is no practical use for such accuracy. With just 15 decimal places, the circumference of a circle with radius equal to the distance between the earth and the sun can be calculated to within just one millimeter. Nobody needs such accuracy.

Nor is there much theoretical interest in the actual decimal expansion of π. The sequence of digits passes every test of randomness ever invented.

We should note that Vega's value contains an error in the 127th digit. Vega gives a 4 where there should be an 8, and all digits after that are incorrect.

Yet Vega was a practical man. It does not make sense to think that he would have done so much tedious and precise work for no practical purpose. I feel sure that Vega undertook this project to give credibility to his ten-place logarithm tables. The public certainly would have believed that, to a man who could calculate accurately to 140 decimal places, surely the ten place calculations would present no great difficulties.

This leads us to ask why anyone would need ten place logarithm tables as well. My younger colleagues may need to be reminded that, before the days of pocket calculators, virtually all calculations had to be done by hand. Certain calculations could be done on slide rules, but they generally gave only two or three decimal place accuracy, and were useful only for multiplication and division and for certain special functions. If people needed more accuracy in multiplication or division problems, then they either performed the tedious calculations by hand, or they used logarithm tables. To multiply two seven-digit numbers by hand was roughly ten times more difficult than the same calculation using logarithm tables, and the effort saved in division was even greater. There were, however, drawbacks to using tables. In general, the number of significant figures in an answer could be no more than the number of decimal places in the tables.

Before Jurij Vega, the most accurate logarithm tables available were the seven place tables calculated by Vlaq and Briggs almost 200 years earlier. Such tables had sufficed for a long time. For most of that time, the very best instruments gave only four or five decimal places of accuracy.

By the 1770's and 1780's, though, instruments were developed that measured seconds of arc accurately. There are 324,000 seconds of arc in 90 degrees, so trigonometric functions of arcs measured to seconds require almost seven significant figures, right at the limits of the tables of the day. A navigator who could measure seconds of arc could calculate his position to within about 30 meters.

In the late 1780's, though, even better instruments were designed and built. Borda, for example, developed a surveying instrument called the Borda Repeating Circle, which measured thirds of arc. A second is 60 thirds, and calculations involving thirds require eight and a half significant figures. The old tables of Briggs and Vlaq were obsolete for the most sophisticated calculations of the day, and the time had come for new ten-place tables.

Vega's ten place tables were accurate enough for almost all practical applications for over 150 years, when they were eventually replaced by electronics.

Vega and Euler

Let us return to Euler's paper published in 1798. We have shown how Vega's work on π was closely related to Euler's. Some of us may worry that Vega only rediscovered something that Euler had done earlier, but had not yet been published. Fortunately, that is not the case, as we will see as we look more closely at Euler's paper.

Euler read "Investigatio quarundam serierum quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae" at the weekly meeting of the St. Petersburg Academy on 7 June 1779. When Euler had returned to St. Petersburg from Berlin in 1766, he had promised to leave enough unpublished papers for the Academy to publish for ten years after his death. In fact, Euler left over 200 papers, and the Academy published them in every volume of their journals, with the exception of the memorial volume for Catherine II, from his death in 1783 until 1830.

This paper, number 705 in the index of Euler's works, was among those Euler intended to be posthumous. Vega's paper, though, made Number 705 timely, so they published it earlier than they might otherwise have done. The paper was inserted into the 1793 volume of the *Nova Acta*, which spent five years in the printer's shop and was published in 1798.

Euler begins his paper with a brief history of π. In 1596, Ludolph van Ceulen (1539-1610) had used a polygon with 60×2^{39} sides to find the value of π to 20 decimal places. Ever since, the number π has often been called the Ludolphine number. We note that Ludolph had something besides an interest in π in common with Jurij Vega; he was professor of mathematics and military science at the University of Leyden.

By Euler's and Vega's time, the record for digits of π had advanced several times. In 1717, De Lagny used the series for $\frac{\pi}{6}=\arctan\frac{1}{\sqrt{3}}$ to find π to 127 decimal places. The series would have been

$$\pi=\sqrt{12}\left(1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\frac{1}{9\cdot 3^4}-...\right)$$

Euler calculates that De Lagny must have evaluated at least 269 terms of this series to get such accuracy.

Euler continues, reestablishing the identities we have seen here several times now, that

$$\begin{aligned}\frac{\pi}{4}&=\arctan\frac{1}{2}+\arctan\frac{1}{3}\\&=2\arctan\frac{1}{3}+\arctan\frac{1}{7}\end{aligned}$$

and that

$$\arctan a=\arctan b+\arctan\frac{a-b}{1+ab}.$$

Euler applies these identities two more times to get two more relations:

$$p = 12\arctan\frac{1}{7} + \arctan\frac{2}{11}$$
$$= 20\arctan\frac{2}{11} + 8\arctan\frac{3}{79}$$

Euler stops applying identities here without telling us why, but we can check his work and find that after this last fraction, $\frac{3}{79}$, the series that arise do not converge as fast as this one does. In all of this, Euler does not give a value of π beyond 31 decimal places. We see that Euler's work here was substantially different from Vega's, and Vega can be cleared of any suspicion that he was just rediscovering Euler's work.

Conclusions

Beckmann derides people who sought to find more and more decimal places of π as "digit hunters." We have seen though, that Vega's record for the digits of π was more than that. His work was based on some good mathematics, and he had solid motivation, in the context of his 10-place logarithm tables and the increasing needs of his time for more and more accurate tables, to demonstrate his skills as a high-precision computer.

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