Euler’s Solution of the Basel Problem – The Longer Story

Most accounts of Euler’s brilliant summing of the reciprocals of the square numbers describe only his final solution to the problem. In fact, the usual solution is only the third of three solutions given in that 1736 paper. We describe some of Euler’s earlier results that led up to the 1736 paper, give all three of the solutions given there, as well as other interesting results in the same paper, and describe some related results that Euler gave later in his career. (Received January 30, 2003)

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An Historical Schema

1. Problems
2. Theories and techniques that solve those problems
3. Problems that resist solution by those techniques

Circles and lines
Euclidean geometry
Appollonian curves

Appollonian curves
Analytic geometry
Subtangents, quadratures

Diophantine problems
Infinite descent
Fermat’s last theorem

Fermat’s Last Theorem
L-functions, modular forms and elliptic curves
???

Tangents, areas and series
Calculus
Basel problem

1. Basel Problem
2. Euler’s methods
3. ???

Not all unsolved problems become important unsolved problems.

1644 Pietro Mengoli (1625-1686)

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\sum_{n=1}^{\infty}\frac{1}{n^2}\,.$$

Euler began with his very first letter to Goldbach, 13 October 1729.

> "Most Celebrated Sir: I have been thinking about the laws by which a series may be interpolated. … The most Celebrated Bernoulli suggested that I write to you."

General term of the "series"of factorial numbers, 1, 2, 6, 24, 120, etc. given by

$$\frac{1\cdot 2^m}{1+m}\cdot\frac{2^{1-m}\cdot 3^m}{2+m}\cdot\frac{3^{1-m}\cdot 4^m}{3+m}\cdot\frac{4^{1-m}\cdot 5^m}{5+m}\ etc.$$

Similarly, interpolates the n^th^ partial sum of the harmonic series,

$$\int_0^1 \frac{1-t^{n-1}}{1-t}\,dt$$

then the n^th^ partial sum of the Basel series

$$\int_0^1 \frac{dx}{x}\left(\int_0^x \frac{1-t^{n-1}}{1-t}\,dt \right).$$

Limit as n goes to infinity gives Basel Problem

Euler-Maclauren approximations to get the integral to six decimal places, 1.644924.

probably recognized this value as

$$\frac{\pi^2}{6}.$$

Didn’t share, so he had a valuable advantage as he raced to solve the problem; he knew the answer.

“Usual” 3rd solution

In Dunham, *Euler The Master of Us All*.

Assume, because they have the same roots,

$$\frac{\sin x}{x}=\left(1-\frac{x}{p}\right)\left(1+\frac{x}{p}\right)\left(1-\frac{x}{2p}\right)\left(1+\frac{x}{2p}\right)\left(1-\frac{x}{3p}\right)\left(1+\frac{x}{3p}\right)\ldots.$$

Taylor series expansion of $\frac{\sin x}{x}$ at $x = 0$ is

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots$$

Matching coefficients,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\frac{p^2}{6}$$

$e^x\frac{\sin x}{x}$ has same roots

First solution

Facts about series

$$a = a + b + c + d + e + f + etc.$$

b “the sum of these terms taken two at a time” without repetition

$$b = ab + ac + bc + ad + bc + ae + bd + af + be + cd + etc.$$

then (ignoring fatal issues of convergence)

$$a^2 + b^2 + c^2 + d^2 + etc. = a^2 - 2b$$

$a, b, g, d, etc.$ sums of terms taken one, two, three and four at a time, respectively

P, *Q*, *R*, *S*, *etc.* sums of terms taken to the first, second, third and fourth powers respectively. Then

$$P = a$$

$$Q = Pa - 2b$$

$$R = Qa - Pb + 3g$$

$$S = Ra - Qb + Pg - 4d, \ etc.$$

s an arc

$y = \sin s$.

Taylor series

$$y = s - \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{s^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + etc.$$

Divide by *y* and subtract 1

$$0 = 1 - \frac{s}{y} + \frac{s^3}{1\cdot 2\cdot 3y} - \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5y} + etc.$$

Factor like a polynomial

$$1 - \frac{s}{y} + \frac{s^3}{1\cdot 2\cdot 3y} - \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5y} + etc. = \left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{B}\right)\left(1 - \frac{s}{C}\right)\left(1 - \frac{s}{D}\right) etc.$$

Match terms containing *s* (α)

$$\frac{1}{y} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} + etc.$$

Match terms s^2 gives β. Similarly s^3 up to s^5

Other arcs with the same sine as *s* are

$$A,\ p - A,\ 2p + A,\ 3p - A,\ 4p + A,\ 5p - A,\ 6p + A,\ etc.$$

$$-p - A,\ -2p + A,\ -3p - A,\ -4p + A,\ -5p - A\ etc.$$

This gives values for *A*, *B*, *C*, etc. so

$$\frac{1}{y}=\frac{1}{A}+\frac{1}{p-A}+\frac{1}{-p-A}+\frac{1}{2p+A}+\frac{1}{-2p+A}+\frac{1}{3p-A}+\frac{1}{-3p+A}$$

Matching coefficients again, he gets

$$a=\frac{1}{y}$$

$$b=0$$

$$g=\frac{-1}{1\cdot 2\cdot 3y}\ etc.$$

so

$$P=\frac{1}{y}$$

$$Q=\frac{P}{y}=\frac{1}{y^2}\ etc.$$

Take $y = 1$, let $q = \frac{p}{2}$. Then

$$\begin{aligned} Q &= \frac{1}{q^2} + \frac{1}{q^2} + \frac{1}{9q^2} + \frac{1}{9q^2} + \frac{1}{25q^2} + \frac{1}{25q^2} + etc. \\ &= \frac{2}{q^2}\left(\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + etc.\right) \end{aligned} .$$

But $Q = \frac{1}{y^2}$ and $y = 1$, so $Q = 1$. So

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + etc. = \frac{q^2}{2} = \frac{p^2}{8}.$$

Sum of the reciprocals of the odd squares is $\frac{p^2}{8}$.

Every even square is a power of 4 times an odd square, so

$$\begin{aligned} 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + etc. &= \left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + etc.\right) \cdot \left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + etc.\right) \\ &= \frac{p^2}{8} \cdot \frac{1}{1 - \frac{1}{4}} \\ &= \frac{p^2}{6} \end{aligned}$$

Second solution takes $y = \frac{\sqrt{3}}{2}$ in place of $y = 1$

Nothing much else with this technique of infinite products, though he tries [E61, E130]

1741 fourth solution

E63, “Demonstration de la somme de cette suite $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+etc.$”

uses only elementary calculus tools,

- Taylor series,
- integration by parts
- generalized binomial theorem

$x=\sin s$, so $s=\arcsin x$ and $ds=\frac{dx}{\sqrt{1-xx}}$.

As integrals,

$$s=\int\frac{dx}{\sqrt{1-xx}}.$$

$$sds=\frac{dx}{\sqrt{1-xx}}\int\frac{dx}{\sqrt{1-xx}}.$$

Integrating x from 0 to 1 gives $\frac{p^2}{8}$

Expand the right using generalized binomial theorem with $m = \frac{-1}{2}$, then integrate

$$s = \int \frac{dx}{\sqrt{1-xx}} = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}x^9 + etc.$$

Multiply by $ds = \frac{dx}{\sqrt{1-xx}}$ gives

$$s\ ds = \frac{x\ dx}{\sqrt{1-xx}} + \frac{1}{2\cdot 3}\frac{x^3\ dx}{\sqrt{1-xx}} + \frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{x^5\ dx}{\sqrt{1-xx}} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{x^7\ dx}{\sqrt{1-xx}} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}\frac{x^9\ dx}{\sqrt{1-xx}} + etc.$$

"One will see clearly, if he gives it proper reflection, that in general,"

$$\int \frac{x^{n+2}\,dx}{\sqrt{1-xx}} = \frac{n+1}{n+2}\int \frac{x^n\,dx}{\sqrt{1-xx}} - \frac{x^{n+1}}{n+2}\sqrt{1-xx}$$

"Proper reflection" means "integration by parts." Integral is taken from 0 to 1, and so the second term on the right drops out, leaving

$$\int \frac{x^{n+2}\,dx}{\sqrt{1-xx}} = \frac{n+1}{n+2}\int \frac{x^n\,dx}{\sqrt{1-xx}}$$

Evaluate the integrals in each term of the integral of *s ds*:

$$\int \frac{xdx}{\sqrt{1-xx}} = 1-\sqrt{1-xx} = 1$$

$$\int \frac{x^3\,dx}{\sqrt{1-xx}} = \frac{2}{3}\int \frac{xdx}{\sqrt{1-xx}} = \frac{2}{3}$$

$$\int \frac{x^5\,dx}{\sqrt{1-xx}} = \frac{4}{5}\int \frac{x^3\,dx}{\sqrt{1-xx}} = \frac{2\cdot 4}{3\cdot 5}$$

$$\int \frac{x^7\,dx}{\sqrt{1-xx}} = \frac{6}{7}\int \frac{x^5\,dx}{\sqrt{1-xx}} = \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}$$

Substitute in the series to get

$$\begin{aligned} \frac{p^2}{8} &= \frac{s^2}{2} \\ &= \int sds \\ &= \int \frac{xdx}{\sqrt{1-xx}} + \frac{1}{2\cdot 3}\int \frac{x^3\,dx}{\sqrt{1-xx}} + \frac{1\cdot 3}{2\cdot 4\cdot 5}\int \frac{x^5\,dx}{\sqrt{1-xx}} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\int \frac{x^7\,dx}{\sqrt{1-xx}} + etc. \end{aligned}$$

Substituting for the integrals

$$\frac{p^2}{8} = 1 + \frac{1}{3\cdot 3} + \frac{1}{5\cdot 5} + \frac{1}{7\cdot 7} + etc.$$

$$\frac{p^{2}}{8} = 1 + \frac{1}{3\cdot 3} + \frac{1}{5\cdot 5} + \frac{1}{7\cdot 7} + etc.$$

As in his first proof, this sum of the reciprocals of the odd squares gives the sum of the reciprocals of all squares to be

$$\frac{p^{2}}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + etc.$$

Why do we use only the third solution?

Why aren’t 1, 2 and 4 “beautiful”?

Basel Problem
Euler’s methods
???

This is a question of literary theory.