MAT 111 Extended Algebra for Business and the Sciences

MAT 112 Algebra for Business and the Sciences

Southern Connecticut State University

6 FACTORING

Figure 6.1 Scientists use factoring to calculate growth rates of infectious diseases such as viruses. (credit: "FotoshopTofs" / Pixabay)

Chapter Outline

Introduction

An epidemic of a disease has broken out. Where did it start? How is it spreading? What can be done to control it? Answers to these and other questions can be found by scientists known as epidemiologists. They collect data and analyze it to study disease and consider possible control measures. Because diseases can spread at alarming rates, these scientists must use their knowledge of mathematics involving factoring. In this chapter, you will learn how to factor and apply factoring to real-life situations.

6.1 Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more expressions
- Factor the greatest common factor from a polynomial
- Factor by grouping

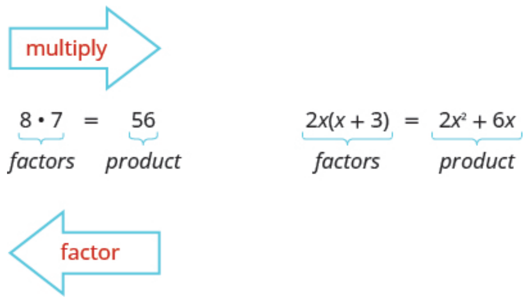
Be Prepared!

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
   If you missed this problem, review Example 1.2.
2. Find the least common multiple (LCM) of 18 and 24.
   If you missed this problem, review Example 1.3.
3. Multiply: $-3a(7a + 8b)$.
   If you missed this problem, review Example 5.26.

Find the Greatest Common Factor of Two or More Expressions

Earlier we multiplied factors together to get a product. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called **factoring**.



We have learned how to factor numbers to find the least common multiple (LCM) of two or more numbers. Now we will factor expressions and find the **greatest common factor** of two or more expressions. The method we use is similar to what we used to find the LCM.

Greatest Common Factor

The **greatest common factor** (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

We summarize the steps we use to find the greatest common factor.

HOW TO : : FIND THE GREATEST COMMON FACTOR (GCF) OF TWO EXPRESSIONS.

Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.

Step 3. Bring down the common factors that all expressions share.

Step 4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three expressions.

EXAMPLE 6.1

Find the greatest common factor of $21x^3, 9x^2, 15x.$

Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
	The GCF of $21x^3$, $9x^2$ and $15x$ is $3x$.

TRY IT : : 6.1 Find the greatest common factor: $25m^4, 35m^3, 20m^2.$

TRY IT : : 6.2 Find the greatest common factor: $14x^3, 70x^2, 105x.$

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4.$ In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x,$ and end with its factors, $3x(x+5)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Distributive Property

If *a*, *b*, and *c* are real numbers, then

$$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

EXAMPLE 6.2 HOW TO USE THE DISTRIBUTIVE PROPERTY TO FACTOR A POLYNOMIAL

Factor: $8m^3 - 12m^2n + 20mn^2$.

Solution

Step 1. Find the GCF of all the terms of the polynomial.	Find the GCF of $8m^3$, $12m^2n$, $20mn^2$	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Step 2. Rewrite each term as a product using the GCF.	Rewrite $8m^3$, $12m^2n$, $20mn^2$ as products of their GCF, $4m$. $8m^3 = 4m \cdot 2m^2$ $12m^2n = -4m \cdot 3m\,n$ $20mn^2 = 4m \cdot 5n^2$	$8m^3 - 12m^2n + 20mn^2$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$
Step 3. Use the "reverse" Distributive Property to factor the expression.		$4m(2m^2 - 3m\,n + 5n^2)$
Step 4. Check by multiplying the factors.		$4m(2m^2 - 3m\,n + 5n^2)$ $4m \cdot 2m^2 - 4m \cdot 3m\,n + 4m \cdot 5n^2$ $8m^3 - 12m^2n + 20mn^2$ ✓

TRY IT : : 6.3 Factor: $9xy^2 + 6x^2y^2 + 21y^3$.

TRY IT : : 6.4 Factor: $3p^3 - 6p^2q + 9pq^3$.

HOW TO : : FACTOR THE GREATEST COMMON FACTOR FROM A POLYNOMIAL.

Step 1. Find the GCF of all the terms of the polynomial.
Step 2. Rewrite each term as a product using the GCF.
Step 3. Use the "reverse" Distributive Property to factor the expression.
Step 4. Check by multiplying the factors.

Factor as a Noun and a Verb

We use “factor” as both a noun and a verb:

Noun:	7 is a *factor* of 14
Verb:	*factor* 3 from $3a + 3$

EXAMPLE 6.3

Factor: $5x^3 - 25x^2$.

Solution

Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $\text{GCF} = 5 \cdot x \cdot x$
	$\text{GCF} = 5x^2$
	$5x^3 - 25x^2$
Rewrite each term.	$5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$5x^2(x - 5)$

Check:

$$5x^2(x - 5)$$
$$5x^2 \cdot x - 5x^2 \cdot 5$$
$$5x^3 - 25x^2 \checkmark$$

TRY IT : : 6.5 Factor: $2x^3 + 12x^2$.

TRY IT : : 6.6 Factor: $6y^3 - 15y^2$.

EXAMPLE 6.4

Factor: $8x^3y - 10x^2y^2 + 12xy^3$.

Solution

The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.	$8x^3y = 2\cdot2\cdot2\cdot x\cdot x\cdot x\cdot y$ $10x^2y^2 = 2\cdot5\cdot x\cdot x\cdot y\cdot y$ $12xy^3 = 2\cdot2\cdot3\cdot x\cdot y\cdot y\cdot y$ $\text{GCF} = 2\cdot x\cdot y$
	$\text{GCF} = 2xy$
	$8x^3y - 10x^2y^2 + 12xy^3$
Rewrite each term using the GCF, $2xy$.	$2xy\cdot4x^2 - 2xy\cdot5xy + 2xy\cdot6y^2$
Factor the GCF.	$2xy(4x^2 - 5xy + 6y^2)$

Check:

$$2xy\left(4x^2 - 5xy + 6y^2\right)$$
$$2xy\cdot4x^2 - 2xy\cdot5xy + 2xy\cdot6y^2$$
$$8x^3y - 10x^2y^2 + 12xy^3\ ✓$$

TRY IT : : 6.7 Factor: $15x^3y - 3x^2y^2 + 6xy^3$.

TRY IT : : 6.8 Factor: $8a^3b + 2a^2b^2 - 6ab^3$.

When the leading coefficient is negative, we factor the negative out as part of the GCF.

EXAMPLE 6.5

Factor: $-4a^3 + 36a^2 - 8a$.

Solution

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3 + 36a^2 - 8a$
Rewrite each term using the GCF, $-4a$.	$-4a\cdot a^2 - (-4a)\cdot9a + (-4a)\cdot2$
Factor the GCF.	$-4a(a^2 - 9a + 2)$

Check:

$$-4a\left(a^2 - 9a + 2\right)$$
$$-4a\cdot a^2 - (-4a)\cdot9a + (-4a)\cdot2$$
$$-4a^3 + 36a^2 - 8a\ ✓$$

TRY IT : : 6.9 Factor: $-4b^3 + 16b^2 - 8b$.

TRY IT : : 6.10 Factor: $-7a^3 + 21a^2 - 14a$.

So far our greatest common factors have been monomials. In the next example, the greatest common factor is a binomial.

EXAMPLE 6.6

Factor: $3y(y+7)-4(y+7)$.

Solution

The GCF is the binomial $y+7$.

	$3y(y+7)-4(y+7)$
Factor the GCF, $(y+7)$.	$(y+7)(3y-4)$
Check on your own by multiplying.	

TRY IT : : 6.11 Factor: $4m(m+3)-7(m+3)$.

TRY IT : : 6.12 Factor: $8n(n-4)+5(n-4)$.

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the GCF in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored. Just like some numbers are prime, some polynomials are prime.

EXAMPLE 6.7 HOW TO FACTOR A POLYNOMIAL BY GROUPING

Factor by grouping: $xy+3y+2x+6$.

Solution

Step 1. Group terms with common factors.	Is there a greatest common factor of all four terms?	$xy+3y+2x+6$
	No, so let's separate the first two terms from the second two.	$\underbrace{xy+3y}+\underbrace{2x+6}$
Step 2. Factor out the common factor in each group.	Factor the GCF from the first two terms.	$y(x+3)+\underbrace{2x+6}$
	Factor the GCF from the second two terms.	$y(x+3)+2(x+3)$
Step 3. Factor the common factor from the expression.	Notice that each term has a common factor of $(x+3)$.	$y(x+3)+2(x+3)$
	Factor out the common factor.	$(x+3)(y+2)$
Step 4. Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $xy+2x+3y+6$ $xy+3y+2x+6$ ✓

TRY IT : : 6.13 Factor by grouping: $xy+8y+3x+24$.

TRY IT : : 6.14 Factor by grouping: $ab+7b+8a+56$.

HOW TO : : FACTOR BY GROUPING.

Step 1. Group terms with common factors.
Step 2. Factor out the common factor in each group.
Step 3. Factor the common factor from the expression.
Step 4. Check by multiplying the factors.

EXAMPLE 6.8

Factor by grouping: ⓐ $x^2 + 3x - 2x - 6$ ⓑ $6x^2 - 3x - 4x + 2$.

Solution

ⓐ

There is no GCF in all four terms.	$x^2 + 3x - 2x - 6$
Separate into two parts.	$x^2 + 3x \quad -2x - 6$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$x(x + 3) - 2(x + 3)$
Factor out the common factor.	$(x + 3)(x - 2)$
Check on your own by multiplying.	

ⓑ

There is no GCF in all four terms.	$6x^2 - 3x - 4x + 2$
Separate into two parts.	$6x^2 - 3x \quad -4x + 2$
Factor the GCF from both parts.	$3x(2x - 1) - 2(2x - 1)$
Factor out the common factor.	$(2x - 1)(3x - 2)$
Check on your own by multiplying.	

TRY IT : : 6.15 Factor by grouping: ⓐ $x^2 + 2x - 5x - 10$ ⓑ $20x^2 - 16x - 15x + 12$.

TRY IT : : 6.16 Factor by grouping: ⓐ $y^2 + 4y - 7y - 28$ ⓑ $42m^2 - 18m - 35m + 15$.

6.1 EXERCISES

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q,\ 12pq^2$

2. $8a^2b^3,\ 10ab^2$

3. $12m^2n^3,\ 30m^5n^3$

4. $28x^2y^4,\ 42x^4y^4$

5. $10a^3,\ 12a^2,\ 14a$

6. $20y^3,\ 28y^2,\ 40y$

7. $35x^3y^2,\ 10x^4y,\ 5x^5y^3$

8. $27p^2q^3,\ 45p^3q^4,\ 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

9. $6m + 9$

10. $14p + 35$

11. $9n - 63$

12. $45b - 18$

13. $3x^2 + 6x - 9$

14. $4y^2 + 8y - 4$

15. $8p^2 + 4p + 2$

16. $10q^2 + 14q + 20$

17. $8y^3 + 16y^2$

18. $12x^3 - 10x$

19. $5x^3 - 15x^2 + 20x$

20. $8m^2 - 40m + 16$

21. $24x^3 - 12x^2 + 15x$

22. $24y^3 - 18y^2 - 30y$

23. $12xy^2 + 18x^2y^2 - 30y^3$

24. $21pq^2 + 35p^2q^2 - 28q^3$

25. $20x^3y - 4x^2y^2 + 12xy^3$

26. $24a^3b + 6a^2b^2 - 18ab^3$

27. $-2x - 4$

28. $-3b + 12$

29. $-2x^3 + 18x^2 - 8x$

30. $-5y^3 + 35y^2 - 15y$

31. $-4p^3q - 12p^2q^2 + 16pq^2$

32. $-6a^3b - 12a^2b^2 + 18ab^2$

33. $5x(x + 1) + 3(x + 1)$

34. $2x(x - 1) + 9(x - 1)$

35. $3b(b - 2) - 13(b - 2)$

36. $6m(m - 5) - 7(m - 5)$

Factor by Grouping

In the following exercises, factor by grouping.

37. $ab + 5a + 3b + 15$

38. $cd + 6c + 4d + 24$

39. $8y^2 + y + 40y + 5$

40. $6y^2 + 7y + 24y + 28$

41. $uv - 9u + 2v - 18$

42. $pq - 10p + 8q - 80$

43. $u^2 - u + 6u - 6$

44. $x^2 - x + 4x - 4$

45. $9p^2 - 3p - 20$

46. $16q^2 - 8q - 35$

47. $mn - 6m - 4n + 24$

48. $r^2 - 3r - r + 3$

49. $2x^2 - 14x - 5x + 35$

50. $4x^2 - 36x - 3x + 27$

Mixed Practice

In the following exercises, factor.

51. $-18xy^2 - 27x^2y$

52. $-4x^3y^5 - x^2y^3 + 12xy^4$

53. $3x^3 - 7x^2 + 6x - 14$

54. $x^3 + x^2 - x - 1$

55. $x^2 + xy + 5x + 5y$

56. $5x^3 - 3x^2 + 5x - 3$

Writing Exercises

57. What does it mean to say a polynomial is in factored form?

58. How do you check result after factoring a polynomial?

59. The greatest common factor of 36 and 60 is 12. Explain what this means.

60. What is the GCF of $y^4, y^5,$ and y^{10}? Write a general rule that tells you how to find the GCF of $y^a, y^b,$ and y^c.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

...with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

6.2 Factor Trinomials

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form $x^2 + bx + c$
- Factor trinomials of the form $ax^2 + bx + c$ using trial and error
- Factor trinomials of the form $ax^2 + bx + c$ using the 'ac' method
- Factor using substitution

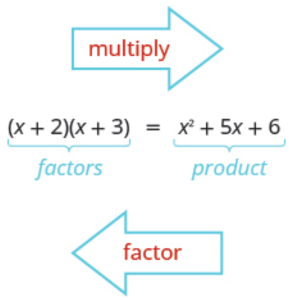
Be Prepared!

Before you get started, take this readiness quiz.

1. Find all the factors of 72.
   If you missed this problem, review **Example 1.2**.
2. Find the product: $(3y + 4)(2y + 5)$.
   If you missed this problem, review **Example 5.28**.
3. Simplify: $-9(6)$; $-9(-6)$.
   If you missed this problem, review **Example 1.18**.

Factor Trinomials of the Form $x^2 + bx + c$

You have already learned how to multiply binomials using FOIL. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a trinomial of the form $x^2 + bx + c$, such as $x^2 + 5x + 6$ and factor it to $(x + 2)(x + 3)$, let's start with two general binomials of the form $(x + m)$ and $(x + n)$.

	$(x + m)(x + n)$
Foil to find the product.	$x^2 + mx + nx + mn$
Factor the GCF from the middle terms.	$x^2 + (m + n)x + mn$
Our trinomial is of the form $x^2 + bx + c$.	$\overbrace{x^2 +}^{x^2 +} \overbrace{(m + n)x}^{bx} \overbrace{+ mn}^{+ c}$

This tells us that to factor a trinomial of the form $x^2 + bx + c$, we need two factors $(x + m)$ and $(x + n)$ where the two numbers *m* and *n* multiply to *c* and add to *b*.

EXAMPLE 6.9 HOW TO FACTOR A TRINOMIAL OF THE FORM $x^2 + bx + c$

Factor: $x^2 + 11x + 24$.

Solution

Step 1. Write the factors as two binomials with first terms *x*.	Write two sets of parentheses and put *x* as the first term.	$x^2 + 11x + 24$ $(x \quad)(x \quad)$
Step 2. Find two numbers *m* and *n* that multiply to *c*, $m \cdot n = c$ add to *b*, $m + n = b$	Find two numbers that multiply to 24 and add to 11.	
Step 3. Use *m* and *n* as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$(x + 3)(x + 8)$
Step 4. Check by multiplying the factors.		$(x + 3)(x + 8)$ $x^2 + 8x + 3x + 24$ $x^2 + 11x + 24$ ✓

Factors of 24	Sum of factors
1, 24	$1 + 24 = 25$
2, 12	$2 + 12 = 14$
3, 8	$3 + 8 = 11$*
4, 6	$4 + 6 = 10$

TRY IT : : 6.17 Factor: $q^2 + 10q + 24$.

TRY IT : : 6.18 Factor: $t^2 + 14t + 24$.

Let's summarize the steps we used to find the factors.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $x^2 + bx + c$.

Step 1. Write the factors as two binomials with first terms *x*. $x^2 + bx + c$ $(x \quad)(x \quad)$

Step 2. Find two numbers *m* and *n* that
- multiply to c, $m \cdot n = c$
- add to b, $m + n = b$

Step 3. Use *m* and *n* as the last terms of the factors. $(x + m)(x + n)$

Step 4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

EXAMPLE 6.10

Factor: $y^2 - 11y + 28$.

Solution

Again, with the positive last term, 28, and the negative middle term, $-11y$, we need two negative factors. Find two numbers that multiply 28 and add to -11.

	$y^2 - 11y + 28$
Write the factors as two binomials with fir t terms y.	$(y \quad)(y \quad)$
Find two numbers that: multiply to 28 and add to -11.	

Factors of 28	Sum of factors
$-1, -28$	$-1 + (-28) = -29$
$-2, -14$	$-2 + (-14) = -16$
$-4, -7$	$-4 + (-7) = -11^*$

Use $-4, -7$ as the last terms of the binomials.	$(y - 4)(y - 7)$

Check:

$$(y - 4)(y - 7)$$
$$y^2 - 7y - 4y + 28$$
$$y^2 - 11y + 28 \checkmark$$

TRY IT : : 6.19 Factor: $u^2 - 9u + 18.$

TRY IT : : 6.20 Factor: $y^2 - 16y + 63.$

Now, what if the last term in the trinomial is negative? Think about FOIL. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

EXAMPLE 6.11

Factor: $2x + x^2 - 48$.

Solution

	$2x + x^2 - 48$
First we put the terms in decreasing degree order.	$x^2 + 2x - 48$
Factors will be two binomials with fir t terms x.	$(x \quad)(x \quad)$

Factors of -48	Sum of factors
$-1, 48$	$-1 + 48 = 47$
$-2, 24$	$-2 + 24 = 22$
$-3, 16$	$-3 + 16 = 13$
$-4, 12$	$-4 + 12 = 8$
$-6, 8$	$-6 + 8 = 2^*$

Use $-6, 8$ as the last terms of the binomials.	$(x - 6)(x + 8)$

Check:

$$(x - 6)(x + 8)$$
$$x^2 - 6q + 8q - 48$$
$$x^2 + 2x - 48 \checkmark$$

TRY IT : : 6.21 Factor: $9m + m^2 + 18$.

TRY IT : : 6.22 Factor: $-7n + 12 + n^2$.

Sometimes you'll need to factor trinomials of the form $x^2 + bxy + cy^2$ with two variables, such as $x^2 + 12xy + 36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, $x \cdot x$. The y^2 in the last term means that the second terms of the binomial factors must each contain *y*. To get the coefficients *b* and *c*, you use the same process summarized in **How To Factor trinomials**.

EXAMPLE 6.12

Factor: $r^2 - 8rs - 9s^2$.

Solution

We need *r* in the first term of each binomial and *s* in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

	$r^2 - 8rs - 9s^2$
Note that the fir t terms are r, last terms contain s.	$(r \quad s)(r \quad s)$
Find the numbers that multiply to -9 and add to -8.	

Factors of -9	Sum of factors
$1, -9$	$-1+9=8$
$-1, 9$	$1+(-9)=-8^*$
$3, -3$	$3+(-3)=0$

Use $1, -9$ as coefficients of he last terms. $(r+s)(r-9s)$

Check:

$$(r-9s)(r+s)$$
$$r^2+rs-9rs-9s^2$$
$$r^2-8rs-9s^2 \checkmark$$

TRY IT : : 6.23 Factor: $a^2-11ab+10b^2$.

TRY IT : : 6.24 Factor: $m^2-13mn+12n^2$.

Some trinomials are prime. The only way to be certain a trinomial is prime is to list all the possibilities and show that none of them work.

EXAMPLE 6.13

Factor: $u^2-9uv-12v^2$.

Solution

We need *u* in the first term of each binomial and *v* in the second term. The last term of the trinomial is negative, so the factors must have opposite signs.

$$u^2-9uv-12v^2$$

Note that the fir t terms are u, last terms contain v. $(u \quad v)(u \quad v)$

Find the numbers that multiply to -12 and add to -9.

Factors of -12	Sum of factors
$1, -12$	$1+(-12)=-11$
$-1, 12$	$-1+12=11$
$2, -6$	$2+(-6)=-4$
$-2, 6$	$-2+6=4$
$3, -4$	$3+(-4)=-1$
$-3, 4$	$-3+4=1$

Note there are no factor pairs that give us -9 as a sum. The trinomial is prime.

TRY IT : : 6.25 Factor: $x^2-7xy-10y^2$.

TRY IT : : 6.26 Factor: $p^2+15pq+20q^2$.

Let’s summarize the method we just developed to factor trinomials of the form x^2+bx+c.

Strategy for Factoring Trinomials of the Form $x^2 + bx + c$

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2 + bx + c$$
$$(x + m)(x + n)$$

When c is positive, m and n have the same sign.

b positive m, n positive	b negative m, n negative
$x^2 + 5x + 6$ $(x + 2)(x + 3)$ same signs	$x^2 - 6x + 8$ $(x - 4)(x - 2)$ same signs

When c is negative, m and n have opposite signs.

$x^2 + x - 12$ $(x + 4)(x - 3)$ opposite signs	$x^2 - 2x - 15$ $(x - 5)(x + 3)$ opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form $ax^2 + bx + c$.

Remember to always check for a GCF first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

EXAMPLE 6.14

Factor completely: $4x^3 + 16x^2 - 20x$.

Solution

Is there a greatest common factor?	$4x^3 + 16x^2 - 20x$
Yes, GCF $= 4x$. Factor it.	$4x\left(x^2 + 4x - 5\right)$
Binomial, trinomial, or more than three terms? It is a trinomial. So "undo FOIL."	$4x(x \quad)(x \quad)$
Use a table like the one shown to find t o numbers that multiply to -5 and add to 4.	$4x(x - 1)(x + 5)$

Factors of -5	Sum of factors
$-1, 5$	$-1 + 5 = 4^*$
$1, -5$	$1 + (-5) = -4$

Check:

$$4x(x - 1)(x + 5)$$
$$4x(x^2 + 5x - x - 5)$$
$$4x(x^2 + 4x - 5)$$
$$4x^3 + 16x^2 - 20x \checkmark$$

TRY IT : : 6.27 Factor completely: $5x^3 + 15x^2 - 20x$.

TRY IT : : 6.28 Factor completely: $6y^3 + 18y^2 - 60y$.

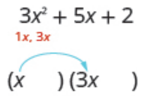
What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2 + 5x + 2$.

From our earlier work, we expect this will factor into two binomials.

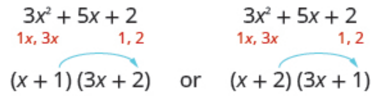
$$\begin{array}{c} 3x^2 + 5x + 2 \\ (\quad\;\;)(\quad\;\;) \end{array}$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x, 3x$. We can place them in the binomials.

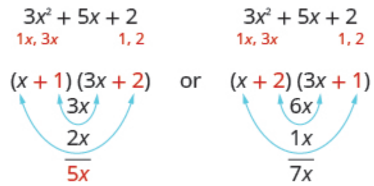


Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1, 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1.



Which factors are correct? To decide that, we multiply the inner and outer terms.



Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$\begin{array}{c} (x + 1)(3x + 2) \\ 3x^2 + 2x + 3x + 2 \\ 3x^2 + 5x + 2\, ✓ \end{array}$$

Our result of the factoring is:

$$\begin{array}{c} 3x^2 + 5x + 2 \\ (x + 1)(3x + 2) \end{array}$$

EXAMPLE 6.15 HOW TO FACTOR A TRINOMIAL USING TRIAL AND ERROR

Factor completely using trial and error: $3y^2 + 22y + 7$.

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$

Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y, 3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ $1y, 3y$ $3y^2 + 22y + 7$ $1y, 3y$ $(y \quad)(3y \quad)$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y \quad)(3y \quad)$
Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y + 1)(3y + 7)$ $3y$ $7y$ $10y$ No! We need $22y$ $3y^2 + 22y + 7$ $1y, 3y \qquad 1, 7$ $(y + 7)(3y + 1)$ $21y$ $+y$ $22y$	$3y^2 + 22y + 7$ **Possible factors** / **Product** $(y + 1)(3y + 7)$ / $3y^2 + 10y + 7$ $(y + 7)(3y + 1)$ / $3y^2 + 22y + 7$
Step 6. Check by multiplying.		$(y + 7)(3y + 1)$ $3y^2 + 22y + 7$ ✓

TRY IT : : 6.29 Factor completely using trial and error: $2a^2 + 5a + 3$.

TRY IT : : 6.30 Factor completely using trial and error: $4b^2 + 5b + 1$.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING TRIAL AND ERROR.

Step 1. Write the trinomial in descending order of degrees as needed.
Step 2. Factor any GCF.
Step 3. Find all the factor pairs of the first term.
Step 4. Find all the factor pairs of the third term.
Step 5. Test all the possible combinations of the factors until the correct product is found.
Step 6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

EXAMPLE 6.16

Factor completely using trial and error: $6b^2 - 13b + 5$.

Solution

The trinomial is already in descending order.	$6b^2 - 13b + 5$
Find the factors of the first term.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$
Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $-1, -5$ $2b \cdot 3b$

Consider all the combinations of factors.

$6b^2 - 13b + 5$	
Possible factors	**Product**
$(b - 1)(6b - 5)$	$6b^2 - 11b + 5$
$(b - 5)(6b - 1)$	$6b^2 - 31b + 5$
$(2b - 1)(3b - 5)$	$6b^2 - 13b + 5$*
$(2b - 5)(3b - 1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $(2b - 1)(3b - 5)$

Check by multiplying:

$$(2b - 1)(3b - 5)$$
$$6b^2 - 10b - 3b + 5$$
$$6b^2 - 13b + 5 \checkmark$$

TRY IT : : 6.31 Factor completely using trial and error: $8x^2 - 13x + 3$.

TRY IT : : 6.32 Factor completely using trial and error: $10y^2 - 37y + 7$.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

EXAMPLE 6.17

Factor completely using trial and error: $18x^2 - 37xy + 15y^2$.

Solution

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $2x \cdot 9x$ $3x \cdot 6x$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $-1, -5$ $2x \cdot 9x$ $-5, -1$ $3x \cdot 6x$

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 3y)(9x - 5y)$

Check by multiplying:

$$(2x - 3y)(9x - 5y)$$
$$18x^2 - 10xy - 27xy + 15y^2$$
$$18x^2 - 37xy + 15y^2 \checkmark$$

TRY IT : : 6.33 Factor completely using trial and error $18x^2 - 3xy - 10y^2$.

TRY IT : : 6.34 Factor completely using trial and error: $30x^2 - 53xy - 21y^2$.

Don’t forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

EXAMPLE 6.18

Factor completely using trial and error: $-10y^4 - 55y^3 - 60y^2$.

Solution

	$-10y^4 - 55y^3 - 60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2 + 11y + 12)$
Factor the trinomial.	$-5y^2\left(\underset{y \cdot 2y}{2y^2} + 11y + \underset{\substack{1 \cdot 12 \\ 2 \cdot 6 \\ 3 \cdot 4}}{12}\right)$

Consider all the combinations.

$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y + 1)(2y + 12)$	Not an option
$(y + 12)(2y + 1)$	$2y^2 + 25y + 12$
$(y + 2)(2y + 6)$	Not an option
$(y + 6)(2y + 2)$	Not an option
$(y + 3)(2y + 4)$	Not an option
$(y + 4)(2y + 3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor $-5y^2$.	$-5y^2(y + 4)(2y + 3)$

Check by multiplying:

$$-5y^2(y + 4)(2y + 3)$$
$$-5y^2\left(2y^2 + 8y + 3y + 12\right)$$
$$-10y^4 - 55y^3 - 60y^2 \checkmark$$

TRY IT : : 6.35 Factor completely using trial and error: $15n^3 - 85n^2 + 100n$.

TRY IT : : 6.36 Factor completely using trial and error: $56q^3 + 320q^2 - 96q$.

Factor Trinomials of the Form $ax^2 + bx + c$ using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

EXAMPLE 6.19 HOW TO FACTOR TRINOMIALS USING THE "AC" METHOD

Factor using the *'ac'* method: $6x^2 + 7x + 2$.

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2 + 7x + 2$
Step 2. Find the product *ac*.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2 + 7x + 2$

Step 3. Find two numbers *m* and *n* that: Multiply to *ac*. $m \cdot n = a \cdot c$ Add to *b*. $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3 + 4 = 7$	
Step 4. Split the middle term using *m* and *n*. $ax^2 + bx + c$ bx $ax^2 + mx + nx + c$	Rewrite $7x$ as $3x + 4x$. It would also give the same result if we used $4x + 3x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $6x^2 + 3x + 4x + 2$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$
Step 6. Check by multiplying the factors.		$(2x + 1)(3x + 2)$ $6x^2 + 4x + 3x + 2$ $6x^2 + 7x + 2$ ✓

TRY IT : : 6.37 Factor using the 'ac' method: $6x^2 + 13x + 2$.

TRY IT : : 6.38 Factor using the 'ac' method: $4y^2 + 8y + 3$.

The "ac" method is summarized here.

HOW TO : : FACTOR TRINOMIALS OF THE FORM $ax^2 + bx + c$ USING THE "AC" METHOD.

Step 1. Factor any GCF.

Step 2. Find the product *ac*.

Step 3. Find two numbers *m* and *n* that:
Multiply to ac $\quad m \cdot n = a \cdot c$
Add to b $\quad m + n = b$
$ax^2 + bx + c$

Step 4. Split the middle term using *m* and *n*. $\quad ax^2 + mx + nx + c$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

Don't forget to look for a common factor!

EXAMPLE 6.20

Factor using the *'ac'* method: $10y^2 - 55y + 70$.

Solution

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2 - 55y + 70$
Factor it.		$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		$ax^2 + bx + c$ $5(2y^2 - 11y + 14)$
Find the product ac.	$ac = 28$	
Find two numbers that multiply to ac	$(-4)(-7) = 28$	
and add to b.	$-4 + (-7) = -11$	
Split the middle term.		$5(2y^2 - 11y + 14)$
		$5(2y^2 - 7y - 4y + 14)$
Factor the trinomial by grouping.		$5(y(2y - 7) - 2(y - 7))$
		$5(y - 2)(2y - 7)$
Check by multiplying all three factors.		

$$5(y-2)(2y-7)$$
$$5\left(2y^2 - 7y - 4y + 14\right)$$
$$5\left(2y^2 - 11y + 14\right)$$
$$10y^2 - 55y + 70 \checkmark$$

TRY IT : : 6.39 Factor using the 'ac' method: $16x^2 - 32x + 12$.

TRY IT : : 6.40 Factor using the 'ac' method: $18w^2 - 39w + 18$.

Factor Using Substitution

Sometimes a trinomial does not appear to be in the $ax^2 + bx + c$ form. However, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c$ form. This is called factoring by substitution. It is standard to use *u* for the substitution.

In the $ax^2 + bx + c$, the middle term has a variable, *x*, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

EXAMPLE 6.21

Factor by substitution: $x^4 - 4x^2 - 5$.

Solution

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$). If we let $u = x^2$, we can put our trinomial in the $ax^2 + bx + c$ form we need to factor it.

	$x^4 - 4x^2 - 5$
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u + 1)(u - 5)$
Replace u with x^2.	$(x^2 + 1)(x^2 - 5)$
Check:	

$$(x^2 + 1)(x^2 - 5)$$
$$x^4 - 5x^2 + x^2 - 5$$
$$x^4 - 4x^2 - 5 \checkmark$$

> **TRY IT : :** 6.41 Factor by substitution: $h^4 + 4h^2 - 12$.

> **TRY IT : :** 6.42 Factor by substitution: $y^4 - y^2 - 20$.

Sometimes the expression to be substituted is not a monomial.

EXAMPLE 6.22

Factor by substitution: $(x - 2)^2 + 7(x - 2) + 12$

Solution

The binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

	$(x - 2)^2 + 7(x - 2) + 12$
Rewrite the trinomial to prepare for the substitution.	$(x - 2)^2 + 7(x - 2) + 12$
Let $u = x - 2$ and substitute.	$u^2 + 7u + 12$
Factor the trinomial.	$(u + 3)(u + 4)$
Replace u with $x - 2$.	$((x - 2) + 3)((x - 2) + 4)$
Simplify inside the parentheses.	$(x + 1)(x + 2)$

This could also be factored by first multiplying out the $(x - 2)^2$ and the $7(x - 2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

> **TRY IT : :** 6.43 Factor by substitution: $(x - 5)^2 + 6(x - 5) + 8$.

> **TRY IT : :** 6.44 Factor by substitution: $(y - 4)^2 + 8(y - 4) + 15$.

MEDIA : :

Access this online resource for additional instruction and practice with factoring.

- **Factor a trinomial using the AC method (https://openstax.org/l/37ACmethod)**

6.2 EXERCISES

Practice Makes Perfect

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

61. $p^2 + 11p + 30$

62. $w^2 + 10x + 21$

63. $n^2 + 19n + 48$

64. $b^2 + 14b + 48$

65. $a^2 + 25a + 100$

66. $u^2 + 101u + 100$

67. $x^2 - 8x + 12$

68. $q^2 - 13q + 36$

69. $y^2 - 18x + 45$

70. $m^2 - 13m + 30$

71. $x^2 - 8x + 7$

72. $y^2 - 5y + 6$

73. $5p - 6 + p^2$

74. $6n - 7 + n^2$

75. $8 - 6x + x^2$

76. $7x + x^2 + 6$

77. $x^2 - 12 - 11x$

78. $-11 - 10x + x^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

79. $x^2 - 2xy - 80y^2$

80. $p^2 - 8pq - 65q^2$

81. $m^2 - 64mn - 65n^2$

82. $p^2 - 2pq - 35q^2$

83. $a^2 + 5ab - 24b^2$

84. $r^2 + 3rs - 28s^2$

85. $x^2 - 3xy - 14y^2$

86. $u^2 - 8uv - 24v^2$

87. $m^2 - 5mn + 30n^2$

88. $c^2 - 7cd + 18d^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

89. $p^3 - 8p^2 - 20p$

90. $q^3 - 5q^2 - 24q$

91. $3m^3 - 21m^2 + 30m$

92. $11n^3 - 55n^2 + 44n$

93. $5x^4 + 10x^3 - 75x^2$

94. $6y^4 + 12y^3 - 48y^2$

95. $2t^2 + 7t + 5$

96. $5y^2 + 16y + 11$

97. $11x^2 + 34x + 3$

98. $7b^2 + 50b + 7$

99. $4w^2 - 5w + 1$

100. $5x^2 - 17x + 6$

101. $4q^2 - 7q - 2$

102. $10y^2 - 53y - 11$

103. $6p^2 - 19pq + 10q^2$

104. $21m^2 - 29mn + 10n^2$

105. $4a^2 + 17ab - 15b^2$

106. $6u^2 + 5uv - 14v^2$

107. $-16x^2 - 32x - 16$

108. $-81a^2 + 153a + 18$

109. $-30q^3 - 140q^2 - 80q$

110. $-5y^3 - 30y^2 + 35y$

Factor Trinomials of the Form $ax^2 + bx + c$ using the 'ac' Method

In the following exercises, factor using the 'ac' method.

111. $5n^2 + 21n + 4$

112. $8w^2 + 25w + 3$

113. $4k^2 - 16k + 15$

114. $5s^2 - 9s + 4$

115. $6y^2 + y - 15$

116. $6p^2 + p - 22$

117. $2n^2 - 27n - 45$

118. $12z^2 - 41z - 11$

119. $60y^2 + 290y - 50$

120. $6u^2 - 46u - 16$

121. $48z^3 - 102z^2 - 45z$

122. $90n^3 + 42n^2 - 216n$

123. $16s^2 + 40s + 24$

124. $24p^2 + 160p + 96$

125. $48y^2 + 12y - 36$

126. $30x^2 + 105x - 60$

Factor Using Substitution

In the following exercises, factor using substitution.

127. $x^4 - x^2 - 12$

128. $x^4 + 2x^2 - 8$

129. $x^4 - 3x^2 - 28$

130. $x^4 - 13x^2 - 30$

131. $(x-3)^2 - 5(x-3) - 36$

132. $(x-2)^2 - 3(x-2) - 54$

133. $(3y-2)^2 - (3y-2) - 2$

134. $(5y-1)^2 - 3(5y-1) - 18$

Mixed Practice

In the following exercises, factor each expression using any method.

135. $u^2 - 12u + 36$

136. $x^2 - 14x - 32$

137. $r^2 - 20rs + 64s^2$

138. $q^2 - 29qr - 96r^2$

139. $12y^2 - 29y + 14$

140. $12x^2 + 36y - 24z$

141. $6n^2 + 5n - 4$

142. $3q^2 + 6q + 2$

143. $13z^2 + 39z - 26$

144. $5r^2 + 25r + 30$

145. $3p^2 + 21p$

146. $7x^2 - 21x$

147. $6r^2 + 30r + 36$

148. $18m^2 + 15m + 3$

149. $24n^2 + 20n + 4$

150. $4a^2 + 5a + 2$

151. $x^4 - 4x^2 - 12$

152. $x^4 - 7x^2 - 8$

153. $(x+3)^2 - 9(x+3) - 36$

154. $(x+2)^2 - 25(x+2) - 54$

Writing Exercises

155. Many trinomials of the form $x^2 + bx + c$ factor into the product of two binomials $(x + m)(x + n)$. Explain how you find the values of *m* and *n*.

156. Tommy factored $x^2 - x - 20$ as $(x + 5)(x - 4)$. Sara factored it as $(x + 4)(x - 5)$. Ernesto factored it as $(x - 5)(x - 4)$. Who is correct? Explain why the other two are wrong.

157. List, in order, all the steps you take when using the "ac" method to factor a trinomial of the form $ax^2 + bx + c$.

158. How is the "ac" method similar to the "undo FOIL" method? How is it different?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form $x^2 + bx + c$			
factor trinomials of the form $ax^2 + bx + c$ using trial and error.			
factor trinomials of the form $ax^2 + bx + c$ with using the "*ac*" method.			
factor using substitution.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

6.3 Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\left(3x^2\right)^3$.
   If you missed this problem, review **Example 5.18**.
2. Multiply: $(m+4)^2$.
   If you missed this problem, review **Example 5.32**.
3. Multiply: $(x-3)(x+3)$.
   If you missed this problem, review **Example 5.33**.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If you learn to recognize these kinds of polynomials, you can use the special products patterns to factor them much more quickly.

Factor Perfect Square Trinomials

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\overset{a\ +\ b}{(3x+4)^2}$$

$$\overset{a^2\ +\ 2\cdot a\ \cdot\ b\ +\ b^2}{(3x)^2 + 2(3x\cdot 4) + 4^2}$$

$$9x^2 + 24x + 16$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, you will start with a perfect square trinomial and factor it into its prime factors.

You could factor this trinomial using the methods described in the last section, since it is of the form ax^2+bx+c. But if you recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, you will save yourself a lot of work.

Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomials Pattern

If *a* and *b* are real numbers

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

To make use of this pattern, you have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, you can easily write the factors.

EXAMPLE 6.23 HOW TO FACTOR PERFECT SQUARE TRINOMIALS

Factor: $9x^2+12x+4$.

Solution

Step 1. Does the trinomial fit the perfect square trinomials pattern, $a^2 + 2ab + b^2$? • Is the first term a perfect square? Write it as a square, a^2. • Is the last term a perfect square? Write it as a square, b^2. • Check the middle term. Is it $2ab$?	Is $9x^2$ a perfect square? Yes—write it as $(3x)^2$. Is 4 a perfect square? Yes—write it as $(2)^2$. Is $12x$ twice the product of $3x$ and 2? Does it match? Yes, so we have a perfect square trinomial!	$9x^2 \quad + \quad 12x \quad + \quad 4$ $(3x)^2$ $(3x)^2 \qquad (2)^2$ $(3x)^2 \searrow 2(3x)(2) \swarrow (2)^2$ $12x$
Step 2. Write the square of the binomial.		$9x^2 + 12x + 4$ $a^2 + 2 \cdot a \cdot b + b^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $(a + b)^2$ $(3x + 2)^2$
Step 3. Check.		$(3x + 2)^2$ $(3x)^2 + 2 \cdot 3x \cdot 2 + 2^2$ $9x^2 + 12x + 4$ ✓

TRY IT : : 6.45 Factor: $4x^2 + 12x + 9.$

TRY IT : : 6.46 Factor: $9y^2 + 24y + 16.$

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2 - 2ab + b^2$, which factors to $(a - b)^2$.

The steps are summarized here.

HOW TO : : FACTOR PERFECT SQUARE TRINOMIALS.

Step 1.	Does the trinomial fit he pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
	Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
	Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
	Check the middle term. Is it $2ab$?	$(a)^2 \searrow 2 \cdot a \cdot b \swarrow (b)^2$	$(a)^2 \searrow 2 \cdot a \cdot b \swarrow (b)^2$
Step 2.	Write the square of the binomial.	$(a + b)^2$	$(a - b)^2$
Step 3.	Check by multiplying.		

We'll work one now where the middle term is negative.

EXAMPLE 6.24

Factor: $81y^2 - 72y + 16$.

Solution

The first and last terms are squares. See if the middle term fits the pattern of a perfect square trinomial. The middle term is negative, so the binomial square would be $(a - b)^2$.

	$81y^2 - 72y + 16$
Are the first and last terms perfect squares?	$(9y)^2$ $(4)^2$
Check the middle term.	$(9y)^2$ $(4)^2$ $2(9y)(4)$ $72y$
Does it match $(a - b)^2$? Yes.	$a^2 - 2 \quad a \quad b + b^2$ $(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$
Write as the square of a binomial.	$(9y - 4)^2$
Check by multiplying:	

$$(9y - 4)^2$$
$$(9y)^2 - 2 \cdot 9y \cdot 4 + 4^2$$
$$81y^2 - 72y + 16 \checkmark$$

TRY IT : : 6.47 Factor: $64y^2 - 80y + 25$.

TRY IT : : 6.48 Factor: $16z^2 - 72z + 81$.

The next example will be a perfect square trinomial with two variables.

EXAMPLE 6.25

Factor: $36x^2 + 84xy + 49y^2$.

Solution

	$36x^2 + 84xy + 49y^2$
Test each term to verify the pattern.	$a^2 + 2 \quad a \quad b + b^2$ $(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$
Factor.	$(6x + 7y)^2$
Check by multiplying.	

$$(6x + 7y)^2$$
$$(6x)^2 + 2 \cdot 6x \cdot 7y + (7y)^2$$
$$36x^2 + 84xy + 49y^2 \checkmark$$

TRY IT : : 6.49 Factor: $49x^2 + 84xy + 36y^2$.

TRY IT : : 6.50 Factor: $64m^2 + 112mn + 49n^2$.

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, you will recognize a perfect square trinomial.

EXAMPLE 6.26

Factor: $100x^2y - 80xy + 16y$.

Solution

	$100x^2y - 80xy + 16y$
Is there a GCF? Yes, $4y$, so factor it out.	$4y(25x^2 - 20x + 4)$
Is this a perfect square trinomial?	
Verify the pattern.	$\overset{a^2 \quad - 2 \quad a \quad b \; + \; b^2}{4y\left[(5x)^2 - 2 \cdot 5x \cdot 2 + 2^2\right]}$
Factor.	$4y(5x - 2)^2$

Remember: Keep the factor $4y$ in the final product.

Check:

$$\begin{array}{c} 4y(5x-2)^2 \\ 4y\left[(5x)^2 - 2 \cdot 5x \cdot 2 + 2^2\right] \\ 4y\left(25x^2 - 20x + 4\right) \\ 100x^2y - 80xy + 16y \checkmark \end{array}$$

TRY IT : : 6.51 Factor: $8x^2y - 24xy + 18y$.

TRY IT : : 6.52 Factor: $27p^2q + 90pq + 75q$.

Factor Differences of Squares

The other special product you saw in the previous chapter was the Product of Conjugates pattern. You used this to multiply two binomials that were conjugates. Here's an example:

$$\overset{(a \; - \; b) \; (a \; + \; b)}{(3x - 4)(3x + 4)}$$

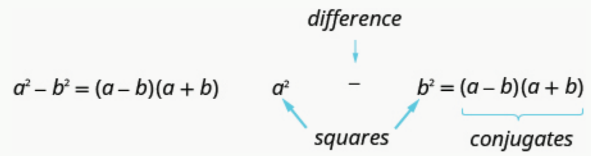
$$\overset{(a)^2 \; - \; (b)^2}{(3x)^2 - (4)^2}$$

$$9x^2 - 16$$

A difference of squares factors to a product of conjugates.

Difference of Squares Pattern

If a and b are real numbers,



Remember, “difference” refers to subtraction. So, to use this pattern you must make sure you have a binomial in which two squares are being subtracted.

EXAMPLE 6.27 HOW TO FACTOR A TRINOMIAL USING THE DIFFERENCE OF SQUARES

Factor: $64y^2 - 1$.

Solution

Step 1. Does the binomial fit the pattern? • Is this a difference? • Are the first and last terms perfect squares?	 Yes Yes	$64y^2 - 1$ $64y^2 - 1$
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$a^2 \quad - \; b^2$ $(8y)^2 - 1^2$
Step 3. Write the product of conjugates.		$(a \; - \; b)\,(a \; + \; b)$ $(8y - 1)(8y + 1)$
Step 4. Check.		$(8y - 1)(8y + 1)$ $64y^2 - 1$ ✓

TRY IT : : 6.53 Factor: $121m^2 - 1$.

TRY IT : : 6.54 Factor: $81y^2 - 1$.

HOW TO : : FACTOR DIFFERENCES OF SQUARES.

Step 1. Does the binomial fit he pattern? $a^2 - b^2$
Is this a diffe ence? $\_\_\_\_ - \_\_\_\_$
Are the fir t and last terms perfect squares?
Step 2. Write them as squares. $(a)^2 - (b)^2$
Step 3. Write the product of conjugates. $(a - b)(a + b)$
Step 4. Check by multiplying.

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ is prime!

The next example shows variables in both terms.

EXAMPLE 6.28

Factor: $144x^2 - 49y^2$.

Solution

	$144x^2 - 49y^2$
Is this a diffe ence of squares? Yes.	$(12x)^2 - (7y)^2$
Factor as the product of conjugates.	$(12x - 7y)(12x + 7y)$
Check by multiplying.	

$$(12x - 7y)(12x + 7y)$$
$$144x^2 - 49y^2 \checkmark$$

> **TRY IT : : 6.55** Factor: $196m^2 - 25n^2$.

> **TRY IT : : 6.56** Factor: $121p^2 - 9q^2$.

As always, you should look for a common factor first whenever you have an expression to factor. Sometimes a common factor may “disguise” the difference of squares and you won’t recognize the perfect squares until you factor the GCF.

Also, to completely factor the binomial in the next example, we’ll factor a difference of squares twice!

EXAMPLE 6.29

Factor: $48x^4y^2 - 243y^2$.

Solution

	$48x^4y^2 - 243y^2$
Is there a GCF? Yes, $3y^2$—factor it out!	$3y^2(16x^4 - 81)$
Is the binomial a diffe ence of squares? Yes.	$3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the fir t binomial is also a diffe ence of squares!	$3y^2\left((2x)^2 - (3)^2\right)(4x^2 + 9)$
Factor it as the product of conjugates.	$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:

$$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$$
$$3y^2(4x^2 - 9)(4x^2 + 9)$$
$$3y^2(16x^4 - 81)$$
$$48x^4y^2 - 243y^2 \checkmark$$

> **TRY IT : : 6.57** Factor: $2x^4y^2 - 32y^2$.

> **TRY IT : : 6.58** Factor: $7a^4c^2 - 7b^4c^2$.

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

EXAMPLE 6.30

Factor: $x^2 - 6x + 9 - y^2$.

Solution

Notice that the first three terms form a perfect square trinomial.

	$x^2 - 6x + 9 - y^2$
Factor by grouping the first three terms.	$\underbrace{x^2 - 6x + 9} - y^2$
Use the perfect square trinomial pattern.	$(x-3)^2 - y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	$\overset{a^2}{(x-3)^2} \overset{-b^2}{- y^2}$
Factor as the product of conjugates.	$\overset{(a \quad - b)}{((x-3) - y)}\overset{(x \quad + b)}{((x-3) + y)}$
	$(x - 3 - y)(x - 3 + y)$

You may want to rewrite the solution as $(x - y - 3)(x + y - 3)$.

> **TRY IT : :** 6.59 Factor: $x^2 - 10x + 25 - y^2$.

> **TRY IT : :** 6.60 Factor: $x^2 + 6x + 9 - 4y^2$.

Factor Sums and Differences of Cubes

There is another special pattern for factoring, one that we did not use when we multiplied polynomials. This is the pattern for the sum and difference of cubes. We will write these formulas first and then check them by multiplication.

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

We'll check the first pattern and leave the second to you.

	$(a + b)(a^2 - ab + b^2)$
Distribute.	$a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$
Multiply.	$a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$
Combine like terms.	$a^3 + b^3$

Sum and Difference of Cubes Pattern

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

The two patterns look very similar, don’t they? But notice the signs in the factors. The sign of the binomial factor matches the sign in the original binomial. And the sign of the middle term of the trinomial factor is the opposite of the sign in the original binomial. If you recognize the pattern of the signs, it may help you memorize the patterns.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

same sign

opposite signs

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

same sign

opposite signs

The trinomial factor in the sum and difference of cubes pattern cannot be factored.

It be very helpful if you learn to recognize the cubes of the integers from 1 to 10, just like you have learned to recognize squares. We have listed the cubes of the integers from 1 to 10 in Table 6.22.

n	1	2	3	4	5	6	7	8	9	10
n^3	1	8	27	64	125	216	343	512	729	1000

Table 6.22

EXAMPLE 6.31 HOW TO FACTOR THE SUM OR DIFFERENCE OF CUBES

Factor: $x^3 + 64$.

Solution

Step 1. Does the binomial fit the sum or difference of cubes pattern? • Is it a sum or difference? • Are the first and last terms perfect cubes?	This is a sum. Yes.	$x^3 + 64$ $x^3 + 64$
Step 2. Write the terms as cubes.	Write them as x^3 and 4^3.	$a^3 + b^3$ $x^3 + 4^3$
Step 3. Use either the sum or difference of cubes pattern.	This is a sum of cubes.	$\binom{a + b}{x + 4}\binom{a^2 - ab + b^2}{x^2 - 4x + 4^2}$
Step 4. Simplify inside the parentheses.	It is already simplified.	$(x + 4)(x^2 - 4x + 16)$
Step 5. Check by multiplying the factors.		$x^2 - 4x + 16$ $x + 4$ $4x^2 - 16x + 64$ $x^3 - 4x^2 + 16x$ $x^3 \quad + 64$ ✓

TRY IT : : 6.61 Factor: $x^3 + 27$.

TRY IT : : 6.62 Factor: $y^3 + 8$.

HOW TO : : FACTOR THE SUM OR DIFFERENCE OF CUBES.

Step 1. Does the binomial fit the sum or difference of cubes pattern?
Is it a sum or difference?
Are the first and last terms perfect cubes?

Step 2. Write them as cubes.

Step 3. Use either the sum or difference of cubes pattern.

Step 4. Simplify inside the parentheses.

Step 5. Check by multiplying the factors.

EXAMPLE 6.32

Factor: $27u^3 - 125v^3$.

Solution

	$27u^3 - 125v^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(3u)^3} \overset{-}{-} \overset{b^3}{(5v)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{3u} \overset{-}{-} \overset{b}{5v}\right)\left(\overset{a^2}{(3u)^2} \overset{+}{+} \overset{ab}{3u \cdot 5v} \overset{+}{+} \overset{b^2}{(5v)^2}\right)$
Simplify.	$\left(\overset{a}{3u} \overset{-}{-} \overset{b}{5v}\right)\left(\overset{a^2}{9u^2} \overset{+}{+} \overset{ab}{15uv} \overset{+}{+} \overset{b^2}{25v^2}\right)$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 6.63 Factor: $8x^3 - 27y^3$.

TRY IT : : 6.64 Factor: $1000m^3 - 125n^3$.

In the next example, we first factor out the GCF. Then we can recognize the sum of cubes.

EXAMPLE 6.33

Factor: $6x^3y + 48y^4$.

Solution

	$6x^3y + 48y^4$
Factor the common factor.	$6y(x^3 + 8y^3)$
This binomial is a sum The first and last terms are perfect cubes.	
Write the terms as cubes.	$6y\left(\overset{a^3\ +}{x^3} + \overset{b^3}{(2y)^3}\right)$
Use the sum of cubes pattern.	$6y\left(\overset{a\ +\ b}{x + 2y}\right)\left(\overset{a^2\ -\ ab\ +\ b^2}{x^2 - x \cdot 2y + (2y)^2}\right)$
Simplify.	$6y(x + 2y)(x^2 - 2xy + 4y^2)$

Check:

To check, you may find it easier to multiply the sum of cubes factors first, then multiply that product by $6y$. We'll leave the multiplication for you.

TRY IT : : 6.65 Factor: $500p^3 + 4q^3$.

TRY IT : : 6.66 Factor: $432c^3 + 686d^3$.

The first term in the next example is a binomial cubed.

EXAMPLE 6.34

Factor: $(x + 5)^3 - 64x^3$.

Solution

	$(x + 5)^3 - 64x^3$
This binomial is a difference. The first and last terms are perfect cubes.	
Write the terms as cubes.	$\overset{a^3}{(x + 5)^3} - \overset{b^3}{(4x)^3}$
Use the difference of cubes pattern.	$\left(\overset{a}{(x + 5)} - \overset{b}{4x}\right)\left(\overset{a^2}{(x + 5)^2} + \overset{a}{(x + 5)} \cdot \overset{b}{4x} + \overset{b^2}{(4x)^2}\right)$
Simplify.	$(x + 5 - 4x)(x^2 + 10x + 25 + 4x^2 + 20x + 16x^2)$
	$(-3x + 5)(21x^2 + 30x + 25)$
Check by multiplying.	We'll leave the check to you.

TRY IT : : 6.67 Factor: $(y + 1)^3 - 27y^3$.

TRY IT : : 6.68 Factor: $(n + 3)^3 - 125n^3$.

MEDIA : :

Access this online resource for additional instruction and practice with factoring special products.

- **Factoring Binomials-Cubes #2 (https://openstax.org/l/37BinomCubes)**

6.3 EXERCISES

Practice Makes Perfect

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

159. $16y^2 + 24y + 9$

160. $25v^2 + 20v + 4$

161. $36s^2 + 84s + 49$

162. $49s^2 + 154s + 121$

163. $100x^2 - 20x + 1$

164. $64z^2 - 16z + 1$

165. $25n^2 - 120n + 144$

166. $4p^2 - 52p + 169$

167. $49x^2 + 28xy + 4y^2$

168. $25r^2 + 60rs + 36s^2$

169. $100y^2 - 52y + 1$

170. $64m^2 - 34m + 1$

171. $10jk^2 + 80jk + 160j$

172. $64x^2y - 96xy + 36y$

173. $75u^4 - 30u^3v + 3u^2v^2$

174. $90p^4 + 300p^4q + 250p^2q^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

175. $25v^2 - 1$

176. $169q^2 - 1$

177. $4 - 49x^2$

178. $121 - 25s^2$

179. $6p^2q^2 - 54p^2$

180. $98r^3 - 72r$

181. $24p^2 + 54$

182. $20b^2 + 140$

183. $121x^2 - 144y^2$

184. $49x^2 - 81y^2$

185. $169c^2 - 36d^2$

186. $36p^2 - 49q^2$

187. $16z^4 - 1$

188. $m^4 - n^4$

189. $162a^4b^2 - 32b^2$

190. $48m^4n^2 - 243n^2$

191. $x^2 - 16x + 64 - y^2$

192. $p^2 + 14p + 49 - q^2$

193. $a^2 + 6a + 9 - 9b^2$

194. $m^2 - 6m + 9 - 16n^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

195. $x^3 + 125$

196. $n^6 + 512$

197. $z^6 - 27$

198. $v^3 - 216$

199. $8 - 343t^3$

200. $125 - 27w^3$

201. $8y^3 - 125z^3$

202. $27x^3 - 64y^3$

203. $216a^3 + 125b^3$

204. $27y^3 + 8z^3$

205. $7k^3 + 56$

206. $6x^3 - 48y^3$

207. $2x^2 - 16x^2y^3$

208. $-2x^3y^2 - 16y^5$

209. $(x+3)^3 + 8x^3$

210. $(x+4)^3 - 27x^3$

211. $(y-5)^3 - 64y^3$

212. $(y-5)^3 + 125y^3$

Mixed Practice

In the following exercises, factor completely.

213. $64a^2 - 25$

214. $121x^2 - 144$

215. $27q^2 - 3$

216. $4p^2 - 100$

217. $16x^2 - 72x + 81$

218. $36y^2 + 12y + 1$

219. $8p^2 + 2$

220. $81x^2 + 169$

221. $125 - 8y^3$

222. $27u^3 + 1000$

223. $45n^2 + 60n + 20$

224. $48q^3 - 24q^2 + 3q$

225. $x^2 - 10x + 25 - y^2$

226. $x^2 + 12x + 36 - y^2$

227. $(x+1)^3 + 8x^3$

228. $(y-3)^3 - 64y^3$

Writing Exercises

229. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

230. How do you recognize the binomial squares pattern?

231. Explain why $n^2 + 25 \neq (n+5)^2$. Use algebra, words, or pictures.

232. Maribel factored $y^2 - 30y + 81$ as $(y-9)^2$. Was she right or wrong? How do you know?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
factor perfect square trinomials.			
factor differences of squares.			
factor sums and differences of cubes.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

6.5 Polynomial Equations

Learning Objectives

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve equations with polynomial functions
- Solve applications modeled by polynomial equations

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $5y - 3 = 0.$
   If you missed this problem, review Example 2.2.
2. Factor completely: $n^3 - 9n^2 - 22n.$
   If you missed this problem, review Example 3.48.
3. If $f(x) = 8x - 16,$ find $f(3)$ and solve $f(x) = 0.$
   If you missed this problem, review Example 3.59.

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

A **polynomial equation** is an equation that contains a polynomial expression. The **degree of the polynomial equation** is the degree of the polynomial.

Polynomial Equation

A **polynomial equation** is an equation that contains a polynomial expression.

The **degree of the polynomial equation** is the degree of the polynomial.

We have already solved polynomial equations of degree one. Polynomial equations of degree one are linear equations are of the form $ax + b = c.$

We are now going to solve polynomial equations of degree two. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2 + 5x + 6 = 0 \qquad 3y^2 + 4y = 10 \qquad 64u^2 - 81 = 0 \qquad n(n + 1) = 42$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get $n^2 + n.$

The general form of a quadratic equation is $ax^2 + bx + c = 0,$ with $a \neq 0.$ (If $a = 0,$ then $0 \cdot x^2 = 0$ and we are left with no quadratic term.)

Quadratic Equation

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a,\ b,\ \text{and } c \text{ are real numbers and } a \neq 0$$

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Product Property

If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.

We will now use the Zero Product Property, to solve a quadratic equation.

EXAMPLE 6.44 HOW TO SOLVE A QUADRATIC EQUATION USING THE ZERO PRODUCT PROPERTY

Solve: $(5n - 2)(6n - 1) = 0$.

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n - 2)(6n - 1) = 0$ $5n - 2 = 0$ or $6n - 1 = 0$
Step 2. Solve the linear equations.	Solve each equation.	$n = \frac{2}{5}$ $\quad n = \frac{1}{6}$
Step 3. Check.	Substitute each solution separately into the original equation.	$n = \frac{2}{5}$ $(5n - 2)(6n - 1) = 0$ $\left(5 \cdot \frac{2}{5} - 2\right)\left(6 \cdot \frac{2}{5} - 1\right) \stackrel{?}{=} 0$ $(2 - 2)\left(\frac{12}{5} - 1\right) \stackrel{?}{=} 0$ $0 \cdot \frac{7}{5} \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $n = \frac{1}{6}$ $(5n - 2)(6n - 1) = 0$ $\left(5 \cdot \frac{1}{6} - 2\right)\left(6 \cdot \frac{1}{6} - 1\right) \stackrel{?}{=} 0$ $\left(\frac{5}{6} - \frac{12}{6}\right)(1 - 1) \stackrel{?}{=} 0$ $\left(-\frac{7}{6}\right)(0) \stackrel{?}{=} 0$ $0 = 0 \checkmark$

TRY IT : : 6.87 Solve: $(3m - 2)(2m + 1) = 0$.

TRY IT : : 6.88 Solve: $(4p + 3)(4p - 3) = 0$.

HOW TO : : USE THE ZERO PRODUCT PROPERTY.

Step 1. Set each factor equal to zero.
Step 2. Solve the linear equations.
Step 3. Check.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So we be sure to start with the quadratic equation in standard form, $ax^2 + bx + c = 0$. Then we factor the expression on the left.

EXAMPLE 6.45 HOW TO SOLVE A QUADRATIC EQUATION BY FACTORING

Solve: $2y^2 = 13y + 45$.

Solution

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	Write the equation in standard form.	$2y^2 = 13y + 45$ $2y^2 - 13y - 45 = 0$
Step 2. Factor the quadratic expression.	Factor $2y^2 - 13y + 45$ $(2y + 5)(y - 9)$	$(2y + 5)(y - 9) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y + 5 = 0 \quad y - 9 = 0$
Step 4. Solve the linear equations.		$y = -\frac{5}{2} \qquad y = 9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y = -\frac{5}{2}$ $2y^2 = 13y + 45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$ $\frac{25}{2} = \frac{25}{2}$ ✓ $y = 9$ $2y^2 = 13y + 45$ $2(9)^2 \stackrel{?}{=} 13(9) + 45$ $2(81) \stackrel{?}{=} 117 + 45$ $162 = 162$ ✓

TRY IT : : 6.89 Solve: $3c^2 = 10c - 8$.

TRY IT : : 6.90 Solve: $2d^2 - 5d = 3$.

HOW TO : : SOLVE A QUADRATIC EQUATION BY FACTORING.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
Step 2. Factor the quadratic expression.
Step 3. Use the Zero Product Property.
Step 4. Solve the linear equations.
Step 5. Check. Substitute each solution separately into the original equation.

Before we factor, we must make sure the quadratic equation is in standard form.

Solving quadratic equations by factoring will make use of all the factoring techniques you have learned in this chapter! Do you recognize the special product pattern in the next example?

EXAMPLE 6.46

Solve: $169q^2 = 49$.

Solution

	$169x^2 = 49$
Write the quadratic equation in standard form.	$169x^2 - 49 = 0$
Factor. It is a diffe ence of squares.	$(13x - 7)(13x + 7) = 0$
Use the Zero Product Property to set each factor to 0.	$13x - 7 = 0 \qquad 13x + 7 = 0$
Solve each equation.	$13x = 7 \qquad 13x = -7$
	$x = \frac{7}{13} \qquad x = -\frac{7}{13}$

Check:

We leave the check up to you.

TRY IT : : 6.91 Solve: $25p^2 = 49$.

TRY IT : : 6.92 Solve: $36x^2 = 121$.

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the Zero Product Property, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

EXAMPLE 6.47

Solve: $(3x - 8)(x - 1) = 3x$.

Solution

	$(3x - 8)(x - 1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x - 2)(x - 4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x - 2 = 0 \qquad x - 4 = 0$
Solve each equation.	$3x = 2 \qquad x = 4$
	$x = \frac{2}{3}$
Check your answers.	The check is left to you.

TRY IT : : 6.93 Solve: $(2m + 1)(m + 3) = 12m$.

TRY IT : : 6.94 Solve: $(k + 1)(k - 1) = 8$.

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

EXAMPLE 6.48

Solve: $3x^2 = 12x + 63$.

Solution

	$3x^2 = 12x + 63$
Write the quadratic equation in standard form.	$3x^2 - 12x - 63 = 0$
Factor the greatest common factor fir t.	$3\left(x^2 - 4x - 21\right) = 0$
Factor the trinomial.	$3(x - 7)(x + 3) = 0$
Use the Zero Product Property to set each factor to 0.	$3 \neq 0 \quad x - 7 = 0 \quad x + 3 = 0$
Solve each equation.	$3 \neq 0 \quad x = 7 \quad x = -3$
Check your answers.	The check is left to you.

TRY IT : : 6.95 Solve: $18a^2 - 30 = -33a$.

TRY IT : : 6.96 Solve: $123b = -6 - 60b^2$.

The Zero Product Property also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

EXAMPLE 6.49

Solve: $9m^3 + 100m = 60m^2$.

Solution

	$9m^3 + 100m = 60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3 - 60m^2 + 100m = 0$
Factor the greatest common factor fir t.	$m\left(9m^2 - 60m + 100\right) = 0$
Factor the trinomial.	$m(3m - 10)(3m - 10) = 0$
Use the Zero Product Property to set each factor to 0.	$m = 0 \quad 3m - 10 = 0 \quad 3m - 10 = 0$
Solve each equation.	$m = 0 \quad m = \frac{10}{3} \quad m = \frac{10}{3}$
Check your answers.	The check is left to you.

TRY IT : : 6.97 Solve: $8x^3 = 24x^2 - 18x$.

TRY IT : : 6.98 Solve: $16y^2 = 32y^3 + 2y$.

Solve Equations with Polynomial Functions

As our study of polynomial functions continues, it will often be important to know when the function will have a certain value or what points lie on the graph of the function. Our work with the Zero Product Property will be help us find these answers.

EXAMPLE 6.50

For the function $f(x) = x^2 + 2x - 2$,

ⓐ find *x* when $f(x) = 6$

ⓑ find two points that lie on the graph of the function.

Solution

ⓐ

	$f(x)=x^2+2x-2$
Substitute 6 for $f(x)$.	$6=x^2+2x-2$
Put the quadratic in standard form.	$x^2+2x-8=0$
Factor the trinomial.	$(x+4)(x-2)=0$
Use the zero product property.	$x+4=0$ or $x-2=0$
Solve.	$x=-4$ or $x=2$
Check:	

$$f(x)=x^2+2x-2 \qquad f(x)=x^2+2x-2$$
$$f(-4)=(-4)^2+2(-4)-2 \qquad f(2)=2^2+2\cdot2-2$$
$$f(-4)=16-8-2 \qquad f(2)=4+4-2$$
$$f(-4)=6\ ✓ \qquad f(2)=6\ ✓$$

ⓑ Since $f(-4)=6$ and $f(2)=6,$ the points $(-4,\ 6)$ and $(2,\ 6)$ lie on the graph of the function.

TRY IT : : 6.99 For the function $f(x)=x^2-2x-8,$

ⓐ find *x* when $f(x)=7$

ⓑ Find two points that lie on the graph of the function.

TRY IT : : 6.100 For the function $f(x)=x^2-8x+3,$

ⓐ find *x* when $f(x)=-4$

ⓑ Find two points that lie on the graph of the function.

The Zero Product Property also helps us determine where the function is zero. A value of *x* where the function is 0, is called a **zero of the function**.

Zero of a Function

For any function *f*, if $f(x)=0,$ then *x* is a **zero of the function**.

When $f(x)=0,$ the point $(x,\ 0)$ is a point on the graph. This point is an *x*-intercept of the graph. It is often important to know where the graph of a function crosses the axes. We will see some examples later.

EXAMPLE 6.51

For the function $f(x)=3x^2+10x-8,$ find

ⓐ the zeros of the function,

ⓑ any *x*-intercepts of the graph of the function

ⓒ any *y*-intercepts of the graph of the function

Solution

ⓐ To find the zeros of the function, we need to find when the function value is 0.

	$f(x) = 3x^2 + 10x - 8$
Substitute 0 for $f(x)$.	$0 = 3x^2 + 10x - 8$
Factor the trinomial.	$(x+4)(3x-2) = 0$
Use the zero product property.	$x + 4 = 0 \quad \text{or} \quad 3x - 2 = 0$
Solve.	$x = -4 \quad \text{or} \quad x = \frac{2}{3}$

ⓑ An x-intercept occurs when $y = 0$. Since $f(-4) = 0$ and $f\left(\frac{2}{3}\right) = 0$, the points $(-4, 0)$ and $\left(\frac{2}{3}, 0\right)$ lie on the graph. These points are x-intercepts of the function.

ⓒ A y-intercept occurs when $x = 0$. To find the y-intercepts we need to find $f(0)$.

	$f(x) = 3x^2 + 10x - 8$
Find $f(0)$ by substituting 0 for x.	$f(0) = 3 \cdot 0^2 + 10 \cdot 0 - 8$
Simplify.	$f(0) = -8$

Since $f(0) = -8$, the point $(0, -8)$ lies on the graph. This point is the y-intercept of the function.

TRY IT : : 6.101 For the function $f(x) = 2x^2 - 7x + 5$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

TRY IT : : 6.102 For the function $f(x) = 6x^2 + 13x - 15$, find

ⓐ the zeros of the function
ⓑ any x-intercepts of the graph of the function
ⓒ any y-intercepts of the graph of the function.

Solve Applications Modeled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

HOW TO : : USE A PROBLEM SOLVING STRATEGY TO SOLVE WORD PROBLEMS.

Step 1. **Read** the problem. Make sure all the words and ideas are understood.
Step 2. **Identify** what we are looking for.
Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
Step 5. **Solve** the equation using appropriate algebra techniques.
Step 6. **Check** the answer in the problem and make sure it makes sense.
Step 7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

EXAMPLE 6.52

The product of two consecutive odd integers is 323. Find the integers.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. We are looking for two consecutive integers.

Step 3. Name what we are looking for. Let $n =$ the fir t integer.

$n + 2 =$ next consecutive odd integer

Step 4. Translate into an equation. Restate the problem in a sentence. The product of the two consecutive odd integers is 323.

$$n(n + 2) = 323$$

Step 5. Solve the equation.

$$n^2 + 2n = 323$$

Bring all the terms to one side.

$$n^2 + 2n - 323 = 0$$

Factor the trinomial.

$$(n - 17)(n + 19) = 0$$

Use the Zero Product Property.

$$n - 17 = 0 \qquad n + 19 = 0$$

Solve the equations.

$$n = 17 \qquad n = -19$$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the fir t integer is $n = 17$ then the next odd integer is

$$n + 2$$
$$17 + 2$$
$$19$$
$$17,\ 19$$

If the fir t integer is $n = -19$ then the next odd integer is

$$n + 2$$
$$-19 + 2$$
$$-17$$
$$-17,\ -19$$

Step 6. Check the answer.

The results are consecutive odd integers $17,\ 19$ and $-19,\ -17$.

$17 \cdot 19 = 323$ ✓ $\qquad -19(-17) = 323$ ✓

Both pairs of consecutive integers are solutions.

Step 7. Answer the question. The consecutive integers are 17, 19 and $-19,\ -17$.

> **TRY IT : :** 6.103 The product of two consecutive odd integers is 255. Find the integers.

> **TRY IT : :** 6.104 The product of two consecutive odd integers is 483 Find the integers.

Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

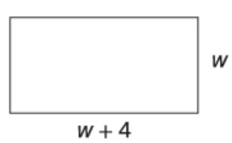
In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

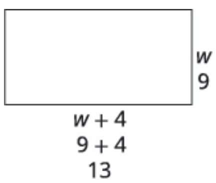
EXAMPLE 6.53

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.



Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A = l \cdot w$
Substitute in the variables.	$117 = (w + 4)w$
Step 5. Solve the equation Distribute first.	$117 = w^2 + 4w$
Get zero on one side.	$117 = w^2 + 4w$
Factor the trinomial.	$0 = w^2 + 4w - 117$
Use the Zero Product Property.	$0 = (w^2 + 13)(w - 9)$
Solve each equation.	$0 = w + 13 \quad 0 = w - 9$
Since *w* is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for *w*.	~~$-13 = w$~~ $\quad 9 = w$
	$w = 9$ Width is 9 feet.
Find the value of the length.	$w + 4$ $9 + 4$ 13 Length is 13 feet.
Step 6. Check the answer. Does the answer make sense?  $A = l \cdot w$ $A = 13 \cdot 9$ $A = 117$ Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

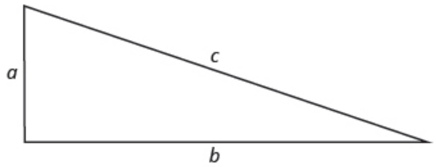
TRY IT : : 6.105

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

TRY IT : : 6.106

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

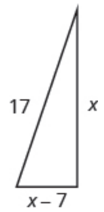
In the next example, we will use the Pythagorean Theorem $\left(a^2 + b^2 = c^2\right)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

EXAMPLE 6.54

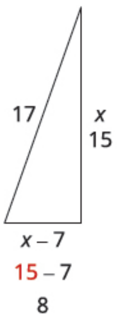
A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Solution

Step 1. Read the problem	
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x =$ length of a side of the sail. $x - 7 =$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$
Step 5. Solve the equation Simplify.	$x^2 + x^2 - 14x + 49 = 289$
	$2x^2 - 14x + 49 = 289$
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$
Factor the trinomial.	$2(x-15)(x+8) = 0$
Use the Zero Product Property.	$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$
Solve.	$2 \neq 0 \quad x = 15 \quad x = -8$
Since *x* is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0 \quad x = 15 \quad \cancel{x = -8}$
Find the length of the other side.	

If the length of one side is then the length of the other side is	$x = 15$ $x - 7$ $15 - 7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense?	



$$a^2 + b^2 = c^2$$

$$15^2 + 8^2 \stackrel{?}{=} 17^2$$

$$225 + 64 \stackrel{?}{=} 289$$

$$289 = 289 \checkmark$$

Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

> **TRY IT : :** 6.107
>
> Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

> **TRY IT : :** 6.108
>
> A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

The next example uses the function that gives the height of an object as a function of time when it is thrown from 80 feet above the ground.

EXAMPLE 6.55

Dennis is going to throw his rubber band ball upward from the top of a campus building. When he throws the rubber band ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tell us when the ball hits the ground

ⓑ when the ball will be 80 feet above the ground

ⓒ the height of the ball at $t = 2$ seconds.

Solution

ⓐ The zeros of this function are found by solving $h(t) = 0$. This will tell us when the ball will hit the ground.

	$h(t) = 0$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 0$
Factor the GCF, -16.	$-16(t^2 - 4t - 5) = 0$
Factor the trinomial.	$-16(t - 5)(t + 1) = 0$
Use the Zero Product Property.	$t - 5 = 0 \qquad t + 1 = 0$
Solve.	$t = 5 \qquad t = -1$

The result $t = 5$ tells us the ball will hit the ground 5 seconds after it is thrown. Since time cannot be negative, the result

$t = -1$ is discarded.

ⓑ The ball will be 80 feet above the ground when $h(t) = 80$.

	$h(t) = 80$
Substitute in the polynomial for $h(t)$.	$-16t^2 + 64t + 80 = 80$
Subtract 80 from both sides.	$-16t^2 + 64t = 0$
Factor the GCF, $-16t$.	$-16t(t - 4) = 0$
Use the Zero Product Property.	$-16t = 0 \quad t - 4 = 0$
Solve.	$t = 0 \quad t = 4$
	The ball will be at 80 feet the moment Dennis tosses the ball and then 4 seconds later, when the ball is falling.

ⓒ To find the height ball at $t = 2$ seconds we find $h(2)$.

	$h(t) = -16t^2 + 64t + 80$
To fin $h(2)$ substitute 2 for t.	$h(2) = -16(2)^2 + 64 \cdot 2 + 80$
Simplify.	$h(2) = 144$
	After 2 seconds, the ball will be at 144 feet.

TRY IT : : 6.109

Genevieve is going to throw a rock from the top a trail overlooking the ocean. When she throws the rock upward from 160 feet above the ocean, the function $h(t) = -16t^2 + 48t + 160$ models the height, *h*, of the rock above the ocean as a function of time, *t*. Find:

ⓐ the zeros of this function which tell us when the rock will hit the ocean
ⓑ when the rock will be 160 feet above the ocean.
ⓒ the height of the rock at $t = 1.5$ seconds.

TRY IT : : 6.110

Calib is going to throw his lucky penny from his balcony on a cruise ship. When he throws the penny upward from 128 feet above the ground, the function $h(t) = -16t^2 + 32t + 128$ models the height, *h*, of the penny above the ocean as a function of time, *t*. Find:

ⓐ the zeros of this function which is when the penny will hit the ocean
ⓑ when the penny will be 128 feet above the ocean.
ⓒ the height the penny will be at $t = 1$ seconds which is when the penny will be at its highest point.

MEDIA : :

Access this online resource for additional instruction and practice with quadratic equations.

- **Beginning Algebra & Solving Quadratics with the Zero Property (https://openstax.org/l/37ZeroProperty)**

6.5 EXERCISES

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

277. $(3a-10)(2a-7)=0$

278. $(5b+1)(6b+1)=0$

279. $6m(12m-5)=0$

280. $2x(6x-3)=0$

281. $(2x-1)^2=0$

282. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

283. $5a^2-26a=24$

284. $4b^2+7b=-3$

285. $4m^2=17m-15$

286. $n^2=5-6n$

287. $7a^2+14a=7a$

288. $12b^2-15b=-9b$

289. $49m^2=144$

290. $625=x^2$

291. $16y^2=81$

292. $64p^2=225$

293. $121n^2=36$

294. $100y^2=9$

295. $(x+6)(x-3)=-8$

296. $(p-5)(p+3)=-7$

297. $(2x+1)(x-3)=-4x$

298. $(y-3)(y+2)=4y$

299. $(3x-2)(x+4)=12x$

300. $(2y-3)(3y-1)=8y$

301. $20x^2-60x=-45$

302. $3y^2-18y=-27$

303. $15x^2-10x=40$

304. $14y^2-77y=-35$

305. $18x^2-9=-21x$

306. $16y^2+12=-32x$

307. $16p^3=24p^2+9p$

308. $m^3-2m^2=-m$

309. $2x^3+72x=24x^2$

310. $3y^3+48y=24y^2$

311. $36x^3+24x^2=-4x$

312. $2y^3+2y^2=12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

313. For the function, $f(x)=x^2-8x+8$, ⓐ find when $f(x)=-4$ ⓑ Use this information to find two points that lie on the graph of the function.

314. For the function, $f(x)=x^2+11x+20$, ⓐ find when $f(x)=-8$ ⓑ Use this information to find two points that lie on the graph of the function.

315. For the function, $f(x)=8x^2-18x+5$, ⓐ find when $f(x)=-4$ ⓑ Use this information to find two points that lie on the graph of the function.

316. For the function, $f(x)=18x^2+15x-10$, ⓐ find when $f(x)=15$ ⓑ Use this information to find two points that lie on the graph of the function.

In the following exercises, for each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph of the function.

317. $f(x)=9x^2-4$

318. $f(x)=25x^2-49$

319. $f(x) = 6x^2 - 7x - 5$

320. $f(x) = 12x^2 - 11x + 2$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

321. The product of two consecutive odd integers is 143. Find the integers.

322. The product of two consecutive odd integers is 195. Find the integers.

323. The product of two consecutive even integers is 168. Find the integers.

324. The product of two consecutive even integers is 288. Find the integers.

325. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

326. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

327. The area of a bulletin board is 55 feet. The length is four feet less than three times the width. Find the length and the width of the a bulletin board.

328. A rectangular carport has area 150 square feet. The height of the carport is five feet less than twice its length. Find the height and the length of the carport.

329. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

330. A stained glass window is shaped like a right triangle. The hypotenuse is 15 feet. One leg is three more than the other. Find the lengths of the legs.

331. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

332. A goat enclosure is in the shape of a right triangle. One leg of the enclosure is built against the side of the barn. The other leg is 4 feet more than the leg against the barn. The hypotenuse is 8 feet more than the leg along the barn. Find the three sides of the goat enclosure.

333. Juli is going to launch a model rocket in her back yard. When she launches the rocket, the function $h(t) = -16t^2 + 32t$ models the height, *h*, of the rocket above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tells us when the penny will hit the ground. ⓑ the time the rocket will be 16 feet above the ground.

334. Gianna is going to throw a ball from the top floor of her middle school. When she throws the ball from 48 feet above the ground, the function $h(t) = -16t^2 + 32t + 48$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find:

ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 48 feet above the ground. ⓒ the height the ball will be at $t = 1$ seconds which is when the ball will be at its highest point.

Writing Exercises

335. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

336. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ *Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

CHAPTER 6 REVIEW

KEY TERMS

degree of the polynomial equation The degree of the polynomial equation is the degree of the polynomial.

factoring Splitting a product into factors is called factoring.

greatest common factor The greatest common factor (GCF) of two or more expressions is the largest expression that is a factor of all the expressions.

polynomial equation A polynomial equation is an equation that contains a polynomial expression.

quadratic equation Polynomial equations of degree two are called quadratic equations.

zero of the function A value of x where the function is 0, is called a zero of the function.

Zero Product Property The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

KEY CONCEPTS

6.1 Greatest Common Factor and Factor by Grouping

- **How to find the greatest common factor (GCF) of two expressions.**
  Step 1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  Step 2. List all factors—matching common factors in a column. In each column, circle the common factors.
  Step 3. Bring down the common factors that all expressions share.
  Step 4. Multiply the factors.
- **Distributive Property:** If *a*, *b*, and *c* are real numbers, then

  $$a(b+c) = ab + ac \quad \text{and} \quad ab + ac = a(b+c)$$

  The form on the left is used to multiply. The form on the right is used to factor.
- **How to factor the greatest common factor from a polynomial.**
  Step 1. Find the GCF of all the terms of the polynomial.
  Step 2. Rewrite each term as a product using the GCF.
  Step 3. Use the “reverse” Distributive Property to factor the expression.
  Step 4. Check by multiplying the factors.
- **Factor as a Noun and a Verb:** We use “factor” as both a noun and a verb.

  $$\begin{array}{ll} \text{Noun:} & 7 \text{ is a } \textit{factor} \text{ of } 14 \\ \text{Verb:} & \textit{factor } 3 \text{ from } 3a+3 \end{array}$$
- **How to factor by grouping.**
  Step 1. Group terms with common factors.
  Step 2. Factor out the common factor in each group.
  Step 3. Factor the common factor from the expression.
  Step 4. Check by multiplying the factors.

6.2 Factor Trinomials

- **How to factor trinomials of the form $x^2 + bx + c$.**
  Step 1. Write the factors as two binomials with first terms *x*. $\quad x^2 + bx + c$ $\quad (x \quad)(x \quad)$
  Step 2. Find two numbers *m* and *n* that

  $$\begin{array}{ll} \text{multiply to} & c,\ m \cdot n = c \\ \text{add to} & b,\ m + n = b \end{array}$$

  Step 3. Use *m* and *n* as the last terms of the factors. $\quad (x+m)(x+n)$

Step 4. Check by multiplying the factors.

- **Strategy for Factoring Trinomials of the Form $x^2 + bx + c$:** When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

$$x^2 + bx + c$$
$$(x + m)(x + n)$$

When c is positive, m and n have the same sign.

b positive	b negative
m, n positive	m, n negative
$x^2 + 5x + 6$	$x^2 - 6x + 8$
$(x + 2)(x + 3)$	$(x - 4)(x - 2)$
same signs	same signs

When c is negative, m and n have opposite signs.

$x^2 + x - 12$	$x^2 - 2x - 15$
$(x + 4)(x - 3)$	$(x - 5)(x + 3)$
opposite signs	opposite signs

Notice that, in the case when *m* and *n* have opposite signs, the sign of the one with the larger absolute value matches the sign of *b*.

- **How to factor trinomials of the form $ax^2 + bx + c$ using trial and error.**

Step 1. Write the trinomial in descending order of degrees as needed.

Step 2. Factor any GCF.

Step 3. Find all the factor pairs of the first term.

Step 4. Find all the factor pairs of the third term.

Step 5. Test all the possible combinations of the factors until the correct product is found.

Step 6. Check by multiplying.

- **How to factor trinomials of the form $ax^2 + bx + c$ using the "ac" method.**

Step 1. Factor any GCF.

Step 2. Find the product *ac*.

Step 3. Find two numbers *m* and *n* that:

Multiply to ac.	$m \cdot n = a \cdot c$
Add to b.	$m + n = b$
	$ax^2 + bx + c$

Step 4. Split the middle term using *m* and *n*. $\quad ax^2 + mx + nx + c$

Step 5. Factor by grouping.

Step 6. Check by multiplying the factors.

6.3 Factor Special Products

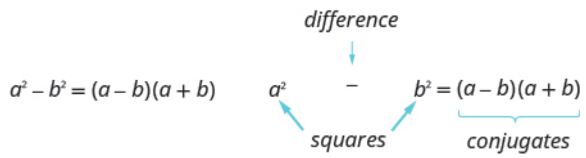
- **Perfect Square Trinomials Pattern:** If *a* and *b* are real numbers,

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

- **How to factor perfect square trinomials.**

Step 1.	Does the trinomial fit he pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
	Is the fir t term a perfect square? Write it as a square.	$(a)^2$	$(a)^2$
	Is the last term a perfect square? Write it as a square.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
	Check the middle term. Is it $2ab$?	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$	$(a)^2 \searrow_{2 \cdot a \cdot b} \swarrow (b)^2$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

- **Difference of Squares Pattern:** If a, b are real numbers,



- **How to factor differences of squares.**

Step 1.	Does the binomial fit he pattern?	$a^2 - b^2$
	Is this a diffe ence?	$\_\_\_\_ - \_\_\_\_$
	Are the fir t and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

- **Sum and Difference of Cubes Pattern**

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

- **How to factor the sum or difference of cubes.**

Step 1. Does the binomial fit the sum or difference of cubes pattern?
Is it a sum or difference?
Are the first and last terms perfect cubes?

Step 2. Write them as cubes.

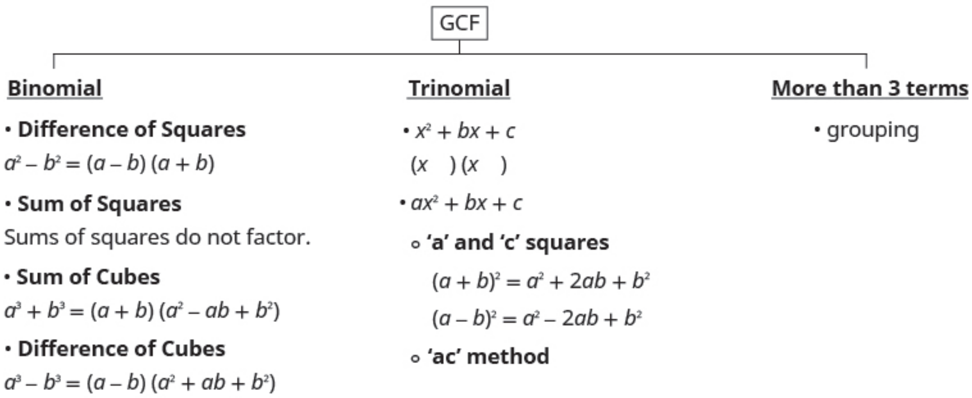
Step 3. Use either the sum or difference of cubes pattern.

Step 4. Simplify inside the parentheses

Step 5. Check by multiplying the factors.

6.4 General Strategy for Factoring Polynomials

General Strategy for Factoring Polynomials



- **How to use a general strategy for factoring polynomials.**

  Step 1. Is there a greatest common factor?
  Factor it out.

  Step 2. Is the polynomial a binomial, trinomial, or are there more than three terms?
  If it is a binomial:
  Is it a sum?
  Of squares? Sums of squares do not factor.
  Of cubes? Use the sum of cubes pattern.
  Is it a difference?
  Of squares? Factor as the product of conjugates.
  Of cubes? Use the difference of cubes pattern.
  If it is a trinomial:
  Is it of the form $x^2 + bx + c$? Undo FOIL.
  Is it of the form $ax^2 + bx + c$?
  If *a* and *c* are squares, check if it fits the trinomial square pattern.
  Use the trial and error or "ac" method.
  If it has more than three terms:
  Use the grouping method.

  Step 3. Check.
  Is it factored completely?
  Do the factors multiply back to the original polynomial?

6.5 Polynomial Equations

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation.

$$a,\ b,\ c \text{ are real numbers and } a \neq 0$$

- **Zero Product Property:** If $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both.
- **How to use the Zero Product Property**

  Step 1. Set each factor equal to zero.

  Step 2. Solve the linear equations.

  Step 3. Check.
- **How to solve a quadratic equation by factoring.**

  Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.

  Step 2. Factor the quadratic expression.

  Step 3. Use the Zero Product Property.

Step 4. Solve the linear equations.

Step 5. Check. Substitute each solution separately into the original equation.

- **Zero of a Function:** For any function *f*, if $f(x) = 0,$ then *x* is a zero of the function.
- **How to use a problem solving strategy to solve word problems.**

Step 1. **Read** the problem. Make sure all the words and ideas are understood.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.

Step 5. **Solve** the equation using appropriate algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

REVIEW EXERCISES

6.1 Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

337. $12a^2b^3,\ 15ab^2$

338. $12m^2n^3,\ 42m^5n^3$

339. $15y^3,\ 21y^2,\ 30y$

340. $45x^3y^2,\ 15x^4y,\ 10x^5y^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

341. $35y + 84$

342. $6y^2 + 12y - 6$

343. $18x^3 - 15x$

344. $15m^4 + 6m^2n$

345. $4x^3 - 12x^2 + 16x$

346. $-3x + 24$

347. $-3x^3 + 27x^2 - 12x$

348. $3x(x - 1) + 5(x - 1)$

Factor by Grouping

In the following exercises, factor by grouping.

349. $ax - ay + bx - by$

350. $x^2y - xy^2 + 2x - 2y$

351. $x^2 + 7x - 3x - 21$

352. $4x^2 - 16x + 3x - 12$

353. $m^3 + m^2 + m + 1$

354. $5x - 5y - y + x$

6.2 Factor Trinomials

Factor Trinomials of the Form $x^2 + bx + c$

In the following exercises, factor each trinomial of the form $x^2 + bx + c$.

355. $a^2 + 14a + 33$

356. $k^2 - 16k + 60$

357. $m^2 + 3m - 54$

358. $x^2 - 3x - 10$

In the following examples, factor each trinomial of the form $x^2 + bxy + cy^2$.

359. $x^2 + 12xy + 35y^2$

360. $r^2 + 3rs - 28s^2$

361. $a^2 + 4ab - 21b^2$

362. $p^2 - 5pq - 36q^2$

363. $m^2 - 5mn + 30n^2$

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

364. $x^3 + 5x^2 - 24x$

365. $3y^3 - 21y^2 + 30y$

366. $5x^4 + 10x^3 - 75x^2$

367. $5y^2 + 14y + 9$

368. $8x^2 + 25x + 3$

369. $10y^2 - 53y - 11$

370. $6p^2 - 19pq + 10q^2$

371. $-81a^2 + 153a + 18$

Factor Trinomials of the Form $ax^2 + bx + c$ using the 'ac' Method

In the following exercises, factor.

372. $2x^2 + 9x + 4$

373. $18a^2 - 9a + 1$

374. $15p^2 + 2p - 8$

375. $15x^2 + 6x - 2$

376. $8a^2 + 32a + 24$

377. $3x^2 + 3x - 36$

378. $48y^2 + 12y - 36$

379. $18a^2 - 57a - 21$

380. $3n^4 - 12n^3 - 96n^2$

Factor using substitution

In the following exercises, factor using substitution.

381. $x^4 - 13x^2 - 30$

382. $(x - 3)^2 - 5(x - 3) - 36$

6.3 Factor Special Products

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

383. $25x^2 + 30x + 9$

384. $36a^2 - 84ab + 49b^2$

385. $40x^2 + 360x + 810$

386. $5k^3 - 70k^2 + 245k$

387. $75u^4 - 30u^3v + 3u^2v^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

388. $81r^2 - 25$

389. $169m^2 - n^2$

390. $25p^2 - 1$

391. $9 - 121y^2$

392. $20x^2 - 125$

393. $169n^3 - n$

394. $6p^2q^2 - 54p^2$

395. $24p^2 + 54$

396. $49x^2 - 81y^2$

397. $16z^4 - 1$

398. $48m^4n^2 - 243n^2$

399. $a^2 + 6a + 9 - 9b^2$

400. $x^2 - 16x + 64 - y^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

401. $a^3 - 125$

402. $b^3 - 216$

403. $2m^3 + 54$

404. $81m^3 + 3$

6.4 General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

405. $24x^3 + 44x^2$

406. $24a^4 - 9a^3$

407. $16n^2 - 56mn + 49m^2$

408. $6a^2 - 25a - 9$

409. $5u^4 - 45u^2$

410. $n^4 - 81$

411. $64j^2 + 225$

412. $5x^2 + 5x - 60$

413. $b^3 - 64$

414. $m^3 + 125$

415. $2b^2 - 2bc + 5cb - 5c^2$

416. $48x^5y^2 - 243xy^2$

417. $5q^2 - 15q - 90$

418. $4u^5v + 4u^2v^3$

419. $10m^4 - 6250$

420. $60x^2y - 75xy + 30y$

421. $16x^2 - 24xy + 9y^2 - 64$

6.5 Polynomial Equations

Use the Zero Product Property

In the following exercises, solve.

422. $(a - 3)(a + 7) = 0$

423. $(5b + 1)(6b + 1) = 0$

424. $6m(12m - 5) = 0$

425. $(2x - 1)^2 = 0$

426. $3m(2m - 5)(m + 6) = 0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

427. $x^2 + 9x + 20 = 0$

428. $y^2 - y - 72 = 0$

429. $2p^2 - 11p = 40$

430. $q^3 + 3q^2 + 2q = 0$

431. $144m^2 - 25 = 0$

432. $4n^2 = 36$

433. $(x + 6)(x - 3) = -8$

434. $(3x - 2)(x + 4) = 12x$

435. $16p^3 = 24p^2 + 9p$

436. $2y^3 + 2y^2 = 12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

437. For the function, $f(x) = x^2 + 11x + 20$, ⓐ find when $f(x) = -8$ ⓑ Use this information to find two points that lie on the graph of the function.

438. For the function, $f(x) = 9x^2 - 18x + 5$, ⓐ find when $f(x) = -3$ ⓑ Use this information to find two points that lie on the graph of the function.

In each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph *of the function.*

439. $f(x) = 64x^2 - 49$

440. $f(x) = 6x^2 - 13x - 5$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

441. The product of two consecutive numbers is 399. Find the numbers.

442. The area of a rectangular shaped patio 432 square feet. The length of the patio is 6 feet more than its width. Find the length and width.

443. A ladder leans against the wall of a building. The length of the ladder is 9 feet longer than the distance of the bottom of the ladder from the building. The distance of the top of the ladder reaches up the side of the building is 7 feet longer than the distance of the bottom of the ladder from the building. Find the lengths of all three sides of the triangle formed by the ladder leaning against the building.

444. Shruti is going to throw a ball from the top of a cliff. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 80 feet above the ground. ⓒ the height the ball will be at $t = 2$ seconds which is when the ball will be at its highest point.

PRACTICE TEST

In the following exercises, factor completely.

445. $80a^2 + 120a^3$

446. $5m(m-1) + 3(m-1)$

447. $x^2 + 13x + 36$

448. $p^2 + pq - 12q^2$

449. $xy - 8y + 7x - 56$

450. $40r^2 + 810$

451. $9s^2 - 12s + 4$

452. $6x^2 - 11x - 10$

453. $3x^2 - 75y^2$

454. $6u^2 + 3u - 18$

455. $x^3 + 125$

456. $32x^5y^2 - 162xy^2$

457. $6x^4 - 19x^2 + 15$

458. $3x^3 - 36x^2 + 108x$

In the following exercises, solve

459. $5a^2 + 26a = 24$

460. The product of two consecutive integers is 156. Find the integers.

461. The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

462. Jing is going to throw a ball from the balcony of her condo. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 128 feet above the ground. ⓒ the height the ball will be at $t = 4$ seconds.

463. For the function, $f(x) = x^2 - 7x + 5,$ ⓐ find when $f(x) = -7$ ⓑ Use this information to find two points that lie on the graph of the function.

464. For the function $f(x) = 25x^2 - 81,$ find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

7 RATIONAL EXPRESSIONS AND FUNCTIONS

Figure 7.1 American football is the most watched spectator sport in the United States. People around the country are constantly tracking statistics for football and other sports. (credit: “keijj44” / Pixabay)

Chapter Outline

Introduction

Twelve goals last season. Fifteen home runs. Nine touchdowns. Whatever the statistics, sports analysts know it. Their jobs depend on it. Compiling and analyzing sports data not only help fans appreciate their teams but also help owners and coaches decide which players to recruit, how to best use them in games, how much they should be paid, and which players to trade. Understanding this kind of data requires a knowledge of specific types of expressions and functions. In this chapter, you will work with rational expressions and perform operations on them. And you will use rational expressions and inequalities to solve real-world problems.

7.1 Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{90y}{15y^2}$.

   If you missed this problem, review **Example 5.13**.

2. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.

   If you missed this problem, review Example 1.25.

3. Divide: $\frac{12}{10} \div \frac{8}{25}$.

   If you missed this problem, review Example 1.26.

We previously reviewed the properties of fractions and their operations. We introduced rational numbers, which are just fractions where the numerators and denominators are integers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Rational Expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

HOW TO : : DETERMINE THE VALUES FOR WHICH A RATIONAL EXPRESSION IS UNDEFINED.

Step 1. Set the denominator equal to zero.

Step 2. Solve the equation.

EXAMPLE 7.1

Determine the value for which each rational expression is undefined:

ⓐ $\frac{8a^2b}{3c}$ ⓑ $\frac{4b-3}{2b+5}$ ⓒ $\frac{x+4}{x^2+5x+6}$.

Solution

The expression will be undefined when the denominator is zero.

ⓐ

$$\frac{8a^2 b}{3c}$$

Set the denominator equal to zero and solve for the variable.

$$3c = 0$$

$$c = 0$$

$$\frac{8a^2 b}{3c} \text{ is undefined or } c = 0.$$

ⓑ

$$\frac{4b-3}{2b+5}$$

Set the denominator equal to zero and solve for the variable.

$$\begin{aligned} 2b+5 &= 0 \\ 2b &= -5 \\ b &= -\frac{5}{2} \end{aligned}$$

$$\frac{4b-3}{2b+5} \text{ is undefined or } b = -\frac{5}{2}.$$

ⓒ

$$\frac{x+4}{x^2+5x+6}$$

Set the denominator equal to zero and solve for the variable.

$$x^2+5x+6 = 0$$

$$(x+2)(x+3) = 0$$

$$x+2 = 0 \text{ or } x+3 = 0$$

$$x = -2 \text{ or } x = -3$$

$$\frac{x+4}{x^2+5x+6} \text{ is undefined or } x = -2 \text{ or } x = -3.$$

TRY IT : : 7.1 Determine the value for which each rational expression is undefined.

ⓐ $\frac{3y^2}{8x}$ ⓑ $\frac{8n-5}{3n+1}$ ⓒ $\frac{a+10}{a^2+4a+3}$

TRY IT : : 7.2 Determine the value for which each rational expression is undefined.

ⓐ $\frac{4p}{5q}$ ⓑ $\frac{y-1}{3y+2}$ ⓒ $\frac{m-5}{m^2+m-6}$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1, in its numerator and denominator.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors in its numerator and denominator.

For example,

$$\frac{x+2}{x+3} \text{ is simplified because here are no common factors of } x+2 \text{ and } x+3.$$

$$\frac{2x}{3x} \text{ is not simplified becaus } x \text{ is a common factor of } 2x \text{ and } 3x.$$

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

If *a*, *b*, and *c* are numbers where $b \neq 0,\ c \neq 0,$

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \quad \text{and} \quad \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0,\ c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$	$\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}}$ where $x \neq 9$	$\frac{x+5}{x}$
$\frac{2}{5}$	$\frac{3x}{5}$	**NO COMMON FACTORS**
We removed the common factors 3 and 7. They are factors of the product.	We removed the common factor $(x-9)$. It is a factor of the product.	While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

EXAMPLE 7.2 HOW TO SIMPLIFY A RATIONAL EXPRESSION

Simplify: $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution

Step 1. Factor the numerator and denominator completely.	Factor x^2+5x+6 and $x^2+8x+12$.	$\frac{x^2+5x+6}{x^2+8x+12}$ $\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Step 2. Simplify by dividing out common factors.	Remove the common factor $x+2$ from the numerator and the denominator.	$\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $\frac{(x+3)}{(x+6)}$ $x \neq -2 \quad x \neq -6$

TRY IT : : 7.3 Simplify: $\frac{x^2-x-2}{x^2-3x+2}$.

TRY IT : : 7.4 Simplify: $\frac{x^2-3x-10}{x^2+x-2}$.

We now summarize the steps you should follow to simplify rational expressions.

HOW TO : : SIMPLIFY A RATIONAL EXPRESSION.

Step 1. Factor the numerator and denominator completely.

Step 2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

EXAMPLE 7.3

Simplify: $\frac{3a^2 - 12ab + 12b^2}{6a^2 - 24b^2}$.

Solution

	$\frac{3a^2 - 12ab + 12b^2}{6a^2 - 24b^2}$
Factor the numerator and denominator, fir t factoring out the GCF.	$\frac{3\left(a^2 - 4ab + 4b^2\right)}{6\left(a^2 - 4b^2\right)}$
	$\frac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$
Remove the common factors of $a - 2b$ and 3.	$\frac{\cancel{3}(a-2b)\cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\cancel{(a-2b)}}$
	$\frac{a-2b}{2(a+2b)}$

TRY IT : : 7.5 Simplify: $\frac{2x^2 - 12xy + 18y^2}{3x^2 - 27y^2}$.

TRY IT : : 7.6 Simplify: $\frac{5x^2 - 30xy + 25y^2}{2x^2 - 50y^2}$.

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a = -1 \cdot a$.

The numerical fraction, say $\frac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\frac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

Let's look at the expression $b - a$.	$b - a$
Rewrite.	$-a + b$
Factor out –1.	$-1(a - b)$

This tells us that $b - a$ is the opposite of $a - b$.

In general, we could write the opposite of $a - b$ as $b - a$. So the rational expression $\frac{a-b}{b-a}$ simplifies to -1.

Opposites in a Rational Expression

The opposite of $a - b$ is $b - a$.

$$\frac{a-b}{b-a} = -1 \quad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a}=1$.

EXAMPLE 7.4

Simplify: $\frac{x^2-4x-32}{64-x^2}$.

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$(-1)\frac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$
Simplify.	$-\frac{x+4}{x+8}$

TRY IT : : 7.7 Simplify: $\frac{x^2-4x-5}{25-x^2}$.

TRY IT : : 7.8 Simplify: $\frac{x^2+x-2}{1-x^2}$.

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If p, q, r, and s are polynomials where $q \neq 0,\ s \neq 0,$ then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0,\ \ x \neq 3,$ and $x \neq 4$.

EXAMPLE 7.5 HOW TO MULTIPLY RATIONAL EXPRESSIONS

Simplify: $\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$ $\frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$

Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$\frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Simplify by dividing out common factors.	Divide out the common factors. Leave the denominator in factored form.	$\frac{\cancel{2}\cancel{x}\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot\cancel{x}\cdot x\cancel{(x-3)}(x-4)}$ $\frac{(x+3)}{3x(x-4)}$

TRY IT : : 7.9 Simplify: $\frac{5x}{x^2+5x+6}\cdot\frac{x^2-4}{10x}$.

TRY IT : : 7.10 Simplify: $\frac{9x^2}{x^2+11x+30}\cdot\frac{x^2-36}{3x^2}$.

HOW TO : : MULTIPLY RATIONAL EXPRESSIONS.

Step 1. Factor each numerator and denominator completely.
Step 2. Multiply the numerators and denominators.
Step 3. Simplify by dividing out common factors.

EXAMPLE 7.6

Multiply: $\frac{3a^2-8a-3}{a^2-25}\cdot\frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

	$\frac{3a^2-8a-3}{a^2-25}\cdot\frac{a^2+10a+25}{3a^2-14a-5}$
Factor the numerators and denominators and then multiply.	$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$
Simplify by dividing out common factors.	$\frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)}$
Simplify.	$\frac{(a-3)(a+5)}{(a-5)(a-5)}$
Rewrite $(a-5)(a-5)$ using an exponent.	$\frac{(a-3)(a+5)}{(a-5)^2}$

TRY IT : : 7.11 Simplify: $\frac{2x^2+5x-12}{x^2-16}\cdot\frac{x^2-8x+16}{2x^2-13x+15}$.

TRY IT : : 7.12 Simplify: $\frac{4b^2+7b-2}{1-b^2}\cdot\frac{b^2-2b+1}{4b^2+15b-4}$.

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

Division of Rational Expressions

If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0,\ r \neq 0,\ s \neq 0,$ then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

EXAMPLE 7.7 HOW TO DIVIDE RATIONAL EXPRESSIONS

Divide: $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$ $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \dfrac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\dfrac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \dfrac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\dfrac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\dfrac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\dfrac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

TRY IT : : 7.13 Simplify: $\dfrac{x^3-8}{3x^2-6x+12} \div \dfrac{x^2-4}{6}$.

TRY IT : : 7.14 Simplify: $\dfrac{2z^2}{z^2-1} \div \dfrac{z^3-z^2+z}{z^3+1}$.

HOW TO : : DIVIDE RATIONAL EXPRESSIONS.

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.

Step 2. Factor the numerators and denominators completely.

Step 3. Multiply the numerators and denominators together.

Step 4. Simplify by dividing out common factors.

Recall from **Use the Language of Algebra** that a complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Also, remember a fraction bar means division. A complex fraction is another way of writing division of two fractions.

EXAMPLE 7.8

Divide: $\dfrac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\dfrac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign. $\dfrac{6x^2-7x+2}{4x-8} \div \dfrac{2x^2-7x+3}{x^2-5x+6}$

Rewrite as product of fir t times reciprocal of second. $\dfrac{6x^2-7x+2}{4x-8} \cdot \dfrac{x^2-5x+6}{2x^2-7x+3}$

Factor the numerators and the denominators, and then multiply. $\dfrac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$

Simplify by dividing out common factors. $\dfrac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$

Simplify. $\dfrac{3x-2}{4}$

TRY IT : : 7.15

Simplify: $\dfrac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

TRY IT : : 7.16

Simplify: $\dfrac{\frac{y^2-36}{2y^2+11y-6}}{\frac{2y^2-2y-60}{8y-4}}$.

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

EXAMPLE 7.9

Perform the indicated operations: $\dfrac{3x-6}{4x-4} \cdot \dfrac{x^2+2x-3}{x^2-3x-10} \div \dfrac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\frac{3(x-2)}{4(x-1)} \cdot \frac{(x+3)(x-1)}{(x+2)(x-5)} \cdot \frac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	$\frac{3\cdot \cancel{8}(x-2)(x+3)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{4}\cdot \cancel{2}\cancel{(x-1)}\cancel{(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

TRY IT : : 7.17 Perform the indicated operations: $\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

TRY IT : : 7.18 Perform the indicated operations: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Multiply and Divide Rational Functions

We started this section stating that a rational expression is an expression of the form $\frac{p}{q}$, where *p* and *q* are polynomials and $q \neq 0$. Similarly, we define a **rational function** as a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

Rational Function

A rational function is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

The domain of a rational function is all real numbers except for those values that would cause division by zero. We must eliminate any values that make $q(x) = 0$.

HOW TO : : DETERMINE THE DOMAIN OF A RATIONAL FUNCTION.

Step 1. Set the denominator equal to zero.
Step 2. Solve the equation.
Step 3. The domain is all real numbers excluding the values found in Step 2.

EXAMPLE 7.10

Find the domain of $R(x) = \frac{2x^2 - 14x}{4x^2 - 16x - 48}$.

Solution

The domain will be all real numbers except those values that make the denominator zero. We will set the denominator equal to zero , solve that equation, and then exclude those values from the domain.

Set the denominator to zero.	$4x^2 - 16x - 48 = 0$
Factor, fir t factor out the GCF.	$4\left(x^2 - 4x - 12\right) = 0$
	$4(x-6)(x+2) = 0$
Use the Zero Product Property.	$4 \neq 0 \quad x - 6 = 0 \quad x + 2 = 0$
Solve.	$x = 6 \quad x = -2$
	The domain of $R(x)$ is all real numbers where $x \neq 6$ and $x \neq -2$.

TRY IT : : 7.19 Find the domain of $R(x) = \frac{2x^2 - 10x}{4x^2 - 16x - 20}$.

TRY IT : : 7.20 Find the domain of $R(x) = \frac{4x^2 - 16x}{8x^2 - 16x - 64}$.

To multiply rational functions, we multiply the resulting rational expressions on the right side of the equation using the same techniques we used to multiply rational expressions.

EXAMPLE 7.11

Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \frac{2x - 6}{x^2 - 8x + 15}$ and $g(x) = \frac{x^2 - 25}{2x + 10}$.

Solution

	$R(x) = f(x) \cdot g(x)$
	$R(x) = \frac{2x - 6}{x^2 - 8x + 15} \cdot \frac{x^2 - 25}{2x + 10}$
Factor each numerator and denominator.	$R(x) = \frac{2(x-3)}{(x-3)(x-5)} \cdot \frac{(x-5)(x+5)}{2(x+5)}$
Multiply the numerators and denominators.	$R(x) = \frac{2(x-3)(x-5)(x+5)}{2(x-3)(x-5)(x+5)}$
Remove common factors.	$R(x) = \frac{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}{\cancel{2}\cancel{(x-3)}\cancel{(x-5)}\cancel{(x+5)}}$
Simplify.	$R(x) = 1$

TRY IT : : 7.21 Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \frac{3x - 21}{x^2 - 9x + 14}$ and $g(x) = \frac{2x^2 - 8}{3x + 6}$.

TRY IT : : 7.22 Find $R(x) = f(x) \cdot g(x)$ where $f(x) = \frac{x^2 - x}{3x^2 + 27x - 30}$ and $g(x) = \frac{x^2 - 100}{x^2 - 10x}$.

To divide rational functions, we divide the resulting rational expressions on the right side of the equation using the same techniques we used to divide rational expressions.

EXAMPLE 7.12

Find $R(x)=\dfrac{f(x)}{g(x)}$ where $f(x)=\dfrac{3x^2}{x^2-4x}$ and $g(x)=\dfrac{9x^2-45x}{x^2-7x+10}$.

Solution

$$R(x)=\frac{f(x)}{g(x)}$$

Substitute in the functions $f(x)$, $g(x)$.

$$R(x)=\frac{\frac{3x^2}{x^2-4x}}{\frac{9x^2-45x}{x^2-7x+10}}$$

Rewrite the division as the product of $f(x)$ and the reciprocal of $g(x)$.

$$R(x)=\frac{3x^2}{x^2-4x}\cdot\frac{x^2-7x+10}{9x^2-45x}$$

Factor the numerators and denominators and then multiply.

$$R(x)=\frac{3\cdot x\cdot x\cdot(x-5)(x-2)}{x(x-4)\cdot 3\cdot 3\cdot x\cdot(x-5)}$$

Simplify by dividing out common factors.

$$R(x)=\frac{\cancel{3}\cdot\cancel{x}\cdot\cancel{x}\cancel{(x-5)}(x-2)}{\cancel{x}(x-4)\cdot\cancel{3}\cdot 3\cdot\cancel{x}\cancel{(x-5)}}$$

$$R(x)=\frac{x-2}{3(x-4)}$$

TRY IT : : 7.23 Find $R(x)=\dfrac{f(x)}{g(x)}$ where $f(x)=\dfrac{2x^2}{x^2-8x}$ and $g(x)=\dfrac{8x^2+24x}{x^2+x-6}$.

TRY IT : : 7.24 Find $R(x)=\dfrac{f(x)}{g(x)}$ where $f(x)=\dfrac{15x^2}{3x^2+33x}$ and $g(x)=\dfrac{5x-5}{x^2+9x-22}$.

7.1 EXERCISES

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1.

ⓐ $\frac{2x^2}{z}$

ⓑ $\frac{4p-1}{6p-5}$

ⓒ $\frac{n-3}{n^2+2n-8}$

2.

ⓐ $\frac{10m}{11n}$

ⓑ $\frac{6y+13}{4y-9}$

ⓒ $\frac{b-8}{b^2-36}$

3.

ⓐ $\frac{4x^2y}{3y}$

ⓑ $\frac{3x-2}{2x+1}$

ⓒ $\frac{u-1}{u^2-3u-28}$

4.

ⓐ $\frac{5pq^2}{9q}$

ⓑ $\frac{7a-4}{3a+5}$

ⓒ $\frac{1}{x^2-4}$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

5. $-\frac{44}{55}$

6. $\frac{56}{63}$

7. $\frac{8m^3n}{12mn^2}$

8. $\frac{36v^3w^2}{27vw^3}$

9. $\frac{8n-96}{3n-36}$

10. $\frac{12p-240}{5p-100}$

11. $\frac{x^2+4x-5}{x^2-2x+1}$

12. $\frac{y^2+3y-4}{y^2-6y+5}$

13. $\frac{a^2-4}{a^2+6a-16}$

14. $\frac{y^2-2y-3}{y^2-9}$

15. $\frac{p^3+3p^2+4p+12}{p^2+p-6}$

16. $\frac{x^3-2x^2-25x+50}{x^2-25}$

17. $\frac{8b^2-32b}{2b^2-6b-80}$

18. $\frac{-5c^2-10c}{-10c^2+30c+100}$

19. $\frac{3m^2+30mn+75n^2}{4m^2-100n^2}$

20. $\frac{5r^2+30rs-35s^2}{r^2-49s^2}$

21. $\frac{a-5}{5-a}$

22. $\frac{5-d}{d-5}$

23. $\frac{20-5y}{y^2-16}$

24. $\frac{4v-32}{64-v^2}$

25. $\frac{w^3+216}{w^2-36}$

26. $\frac{v^3+125}{v^2-25}$

27. $\frac{z^2-9z+20}{16-z^2}$

28. $\frac{a^2-5z-36}{81-a^2}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

29. $\frac{12}{16} \cdot \frac{4}{10}$

30. $\frac{32}{5} \cdot \frac{16}{24}$

31. $\frac{5x^2y^4}{12xy^3} \cdot \frac{6x^2}{20y^2}$

32. $\frac{12a^3b}{b^2} \cdot \frac{2ab^2}{9b^3}$

33. $\frac{5p^2}{p^2 - 5p - 36} \cdot \frac{p^2 - 16}{10p}$

34. $\frac{3q^2}{q^2 + q - 6} \cdot \frac{q^2 - 9}{9q}$

35. $\frac{2y^2 - 10y}{y^2 + 10y + 25} \cdot \frac{y + 5}{6y}$

36. $\frac{z^2 + 3z}{z^2 - 3z - 4} \cdot \frac{z - 4}{z^2}$

37. $\frac{28 - 4b}{3b - 3} \cdot \frac{b^2 + 8b - 9}{b^2 - 49}$

38. $\frac{72m - 12m^2}{8m + 32} \cdot \frac{m^2 + 10m + 24}{m^2 - 36}$

39. $\frac{5c^2 + 9c + 2}{c^2 - 25} \cdot \frac{c^2 + 10c + 25}{3c^2 - 14c - 5}$

40. $\frac{2d^2 + d - 3}{d^2 - 16} \cdot \frac{d^2 - 8d + 16}{2d^2 - 9d - 18}$

41. $\frac{6m^2 - 2m - 10}{9 - m^2} \cdot \frac{m^2 - 6m + 9}{6m^2 + 29m - 20}$

42. $\frac{2n^2 - 3n - 14}{25 - n^2} \cdot \frac{n^2 - 10n + 25}{2n^2 - 13n + 21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

43. $\frac{v - 5}{11 - v} \div \frac{v^2 - 25}{v - 11}$

44. $\frac{10 + w}{w - 8} \div \frac{100 - w^2}{8 - w}$

45. $\frac{3s^2}{s^2 - 16} \div \frac{s^3 - 4s^2 + 16s}{s^3 - 64}$

46. $\frac{r^2 - 9}{15} \div \frac{r^3 - 27}{5r^2 + 15r + 45}$

47. $\frac{p^3 + q^3}{3p^2 + 3pq + 3q^2} \div \frac{p^2 - q^2}{12}$

48. $\frac{v^3 - 8w^3}{2v^2 + 4vw + 8w^2} \div \frac{v^2 - 4w^2}{4}$

49. $\frac{x^2 + 3x - 10}{4x} \div (2x^2 + 20x + 50)$

50. $\frac{2y^2 - 10yz - 48z^2}{2y - 1} \div (4y^2 - 32yz)$

51. $\dfrac{\frac{2a^2 - a - 21}{5a + 20}}{\frac{a^2 + 7a + 12}{a^2 + 8a + 16}}$

52. $\dfrac{\frac{3b^2 + 2b - 8}{12b + 18}}{\frac{3b^2 + 2b - 8}{2b^2 - 7b - 15}}$

53. $\dfrac{\frac{12c^2 - 12}{2c^2 - 3c + 1}}{\frac{4c + 4}{6c^2 - 13c + 5}}$

54. $\dfrac{\frac{4d^2 + 7d - 2}{35d + 10}}{\frac{d^2 - 4}{7d^2 - 12d - 4}}$

For the following exercises, perform the indicated operations.

55. $\dfrac{10m^2+80m}{3m-9} \cdot \dfrac{m^2+4m-21}{m^2-9m+20} \div \dfrac{5m^2+10m}{2m-10}$

56. $\dfrac{4n^2+32n}{3n+2} \cdot \dfrac{3n^2-n-2}{n^2+n-30} \div \dfrac{108n^2-24n}{n+6}$

57. $\dfrac{12p^2+3p}{p+3} \div \dfrac{p^2+2p-63}{p^2-p-12} \cdot \dfrac{p-7}{9p^3-9p^2}$

58. $\dfrac{6q+3}{9q^2-9q} \div \dfrac{q^2+14q+33}{q^2+4q-5} \cdot \dfrac{4q^2+12q}{12q+6}$

Multiply and Divide Rational Functions

In the following exercises, find the domain of each function.

59. $R(x) = \dfrac{x^3-2x^2-25x+50}{x^2-25}$

60. $R(x) = \dfrac{x^3+3x^2-4x-12}{x^2-4}$

61. $R(x) = \dfrac{3x^2+15x}{6x^2+6x-36}$

62. $R(x) = \dfrac{8x^2-32x}{2x^2-6x-80}$

For the following exercises, find $R(x) = f(x) \cdot g(x)$ where $f(x)$ and $g(x)$ are given.

63. $f(x) = \dfrac{6x^2-12x}{x^2+7x-18}$

$g(x) = \dfrac{x^2-81}{3x^2-27x}$

64. $f(x) = \dfrac{x^2-2x}{x^2+6x-16}$

$g(x) = \dfrac{x^2-64}{x^2-8x}$

65. $f(x) = \dfrac{4x}{x^2-3x-10}$

$g(x) = \dfrac{x^2-25}{8x^2}$

66. $f(x) = \dfrac{2x^2+8x}{x^2-9x+20}$

$g(x) = \dfrac{x-5}{x^2}$

For the following exercises, find $R(x) = \dfrac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are given.

67. $f(x) = \dfrac{27x^2}{3x-21}$

$g(x) = \dfrac{3x^2+18x}{x^2+13x+42}$

68. $f(x) = \dfrac{24x^2}{2x-8}$

$g(x) = \dfrac{4x^3+28x^2}{x^2+11x+28}$

69. $f(x) = \dfrac{16x^2}{4x+36}$

$g(x) = \dfrac{4x^2-24x}{x^2+4x-45}$

70. $f(x) = \dfrac{24x^2}{2x-4}$

$g(x) = \dfrac{12x^2+36x}{x^2-11x+18}$

Writing Exercises

71. Explain how you find the values of x for which the rational expression $\dfrac{x^2-x-20}{x^2-4}$ is undefined.

72. Explain all the steps you take to simplify the rational expression $\dfrac{p^2+4p-21}{9-p^2}$.

73. ⓐ Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.

ⓑ Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.

ⓒ Evaluate your answer to part ⓑ when $n = 7$. Did you get the same answer you got in part ⓐ? Why or why not?

74. ⓐ Divide $\frac{24}{5} \div 6$ and explain all your steps.

ⓑ Divide $\frac{x^2 - 1}{x} \div (x + 1)$ and explain all your steps.

ⓒ Evaluate your answer to part ⓑ when $x = 5$. Did you get the same answer you got in part ⓐ? Why or why not?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			
multiply and divide rational functions.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!*

...with some help. *This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.*

7.2 Add and Subtract Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{10} + \frac{8}{15}$.
   If you missed this problem, review **Example 1.29**.
2. Subtract: $\frac{3x}{4} - \frac{8}{9}$.
   If you missed this problem, review **Example 1.28**.
3. Subtract: $6(2x + 1) - 4(x - 5)$.
   If you missed this problem, review **Example 1.56**.

Add and Subtract Rational Expressions with a Common Denominator

What is the first step you take when you add numerical fractions? You check if they have a common denominator. If they do, you add the numerators and place the sum over the common denominator. If they do not have a common denominator, you find one before you add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, you add the numerators and place the sum over the common denominator.

Rational Expression Addition and Subtraction

If *p*, *q*, and *r* are polynomials where $r \neq 0,$ then

$$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r} \quad \text{and} \quad \frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after you subtract the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of *x* should be excluded in the next example?

EXAMPLE 7.13

Add: $\frac{11x + 28}{x + 4} + \frac{x^2}{x + 4}$.

Solution

Since the denominator is $x + 4,$ we must exclude the value $x = -4.$

	$\frac{11x+28}{x+4}+\frac{x^2}{x+4}, \quad x \neq -4$
The fractions have a common denominator, so add the numerators and place the sum over the common denominator.	$\frac{11x+28+x^2}{x+4}$
Write the degrees in descending order.	$\frac{x^2+11x+28}{x+4}$
Factor the numerator.	$\frac{(x+4)(x+7)}{x+4}$
Simplify by removing common factors.	$\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$
Simplify.	$x+7$

The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x \neq -4$.

TRY IT : : 7.25 Simplify: $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

TRY IT : : 7.26 Simplify: $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, you subtract the numerators and place the difference over the common denominator. Be careful of the signs when you subtract a binomial or trinomial.

EXAMPLE 7.14

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

$$\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$$

Subtract the numerators and place the diffe ence over the common denominator.	$\frac{5x^2-7x+3-\left(4x^2+x-9\right)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\cancel{(x-6)}}{(x+3)\cancel{(x-6)}}$
	$\frac{(x-2)}{(x+3)}$

TRY IT : : 7.27 Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

TRY IT : : 7.28 Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d}+\frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d}+\frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$
Simplify.	$\frac{2}{d}$

Be careful with the signs as you work with the opposites when the fractions are being subtracted.

EXAMPLE 7.15

Subtract: $\frac{m^2 - 6m}{m^2 - 1} - \frac{3m + 2}{1 - m^2}$.

Solution

	$\frac{m^2 - 6m}{m^2 - 1} - \frac{3m + 2}{1 - m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2 - 6m}{m^2 - 1} - \frac{-1(3m + 2)}{-1(1 - m^2)}$
Simplify the second fraction.	$\frac{m^2 - 6m}{m^2 - 1} - \frac{-3m - 2}{m^2 - 1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2 - 6m - (-3m - 2)}{m^2 - 1}$
Distribute.	$\frac{m^2 - 6m + 3m + 2}{m^2 - 1}$
Combine like terms.	$\frac{m^2 - 3m + 2}{m^2 - 1}$
Factor the numerator and denominator.	$\frac{(m - 1)(m - 2)}{(m - 1)(m + 1)}$
Simplify by removing common factors.	$\frac{\cancel{(m - 1)}(m - 2)}{\cancel{(m - 1)}(m + 1)}$
Simplify.	$\frac{m - 2}{m + 1}$

TRY IT : : 7.29 Subtract: $\frac{y^2 - 5y}{y^2 - 4} - \frac{6y - 6}{4 - y^2}$.

TRY IT : : 7.30 Subtract: $\frac{2n^2 + 8n - 1}{n^2 - 1} - \frac{n^2 - 7n - 1}{1 - n^2}$.

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12} + \frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12} + \frac{5}{18}$$

$$\frac{7 \cdot 3}{12 \cdot 3} + \frac{5 \cdot 2}{18 \cdot 2}$$

$$\frac{21}{36} + \frac{10}{36}$$

$$\begin{array}{l} 12 = 2 \cdot 2 \cdot 3 \\ 18 = 2 \cdot \phantom{2 \cdot{}} 3 \cdot 3 \\ \hline \text{LCD} = 2 \cdot 2 \cdot 3 \cdot 3 \\ \text{LCD} = 36 \end{array}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

HOW TO : : FIND THE LEAST COMMON DENOMINATOR OF RATIONAL EXPRESSIONS.

Step 1. Factor each denominator completely.
Step 2. List the factors of each denominator. Match factors vertically when possible.
Step 3. Bring down the columns by including all factors, but do not include common factors twice.
Step 4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

EXAMPLE 7.16

ⓐ Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

ⓐ

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	
Factor each denominator completely, lining up common factors.	$x^2-2x-3=(x+1)(x-3)$
Bring down the columns.	$x^2+4x+3=(x+1)\quad(x+3)$ LCD $=(x+1)(x-3)(x+3)$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

ⓑ

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

TRY IT : : 7.31

ⓐ Find the LCD for the expressions $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

TRY IT : : 7.32

ⓐ Find the LCD for the expressions $\frac{3x}{x^2-3x+10}, \frac{5}{x^2+3x+2}$ ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

EXAMPLE 7.17 HOW TO ADD RATIONAL EXPRESSIONS WITH UNLIKE DENOMINATORS

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$.	$x-3:(x-3)$ $x-2:\quad(x-2)$ LCD : $(x-3)(x-2)$
		$\frac{3}{x-3}+\frac{2}{x-2}$
	Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$.	$\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$
	Keep the denominators factored!	$\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

TRY IT : : 7.33 Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

TRY IT : : 7.34 Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

The steps used to add rational expressions are summarized here.

HOW TO : : ADD OR SUBTRACT RATIONAL EXPRESSIONS.

Step 1. Determine if the expressions have a common denominator.
- **Yes** – go to step 2.
- **No** – Rewrite each rational expression with the LCD.
  - Find the LCD.
  - Rewrite each rational expression as an equivalent rational expression with the LCD.

Step 2. Add or subtract the rational expressions.

Step 3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after you have combined the numerators.

EXAMPLE 7.18

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $x^2-2x-3=(x+1)(x-3)$; $x^2+4x+3=(x+1)\quad(x+3)$; LCD $=(x+1)(x-3)(x+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

TRY IT : : 7.35 Add: $\frac{1}{m^2-m-2}+\frac{5m}{m^2+3m+2}$.

TRY IT : : 7.36 Add: $\frac{2n}{n^2-3n-10}+\frac{6}{n^2+5n+6}$.

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

EXAMPLE 7.19

Subtract: $\frac{8y}{y^2-16}-\frac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16} - \frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} y^2 - 16 &= (y-4)(y+4) \\ y - 4 &= y - 4 \\ \text{LCD} &= (y-4)(y+4) \end{aligned}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)} - \frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)} - \frac{4y+16}{(y-4)(y+4)}$
Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

TRY IT : : 7.37 Subtract: $\frac{2x}{x^2-4} - \frac{1}{x+2}$.

TRY IT : : 7.38 Subtract: $\frac{3}{z+3} - \frac{6z}{z^2-9}$.

There are lots of negative signs in the next example. Be extra careful.

EXAMPLE 7.20

Subtract: $\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6} - \frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(-1)(2-n)}$
Write $(-1)(2-n)$ as $n-2$.	$\frac{-3n-9}{(n-2)(n+3)} - \frac{(-1)(n+3)}{(n-2)}$
Simplify. Remember, $a-(-b)=a+b$.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)}{(n-2)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $n^2+n-6=(n-2)(n+3)$; $n-2=(n-2)$; LCD $=(n-2)(n+3)$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)} + \frac{n^2+6n+9}{(n-2)(n+3)}$
Add the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n\cancel{(n+3)}}{(n-2)\cancel{(n+3)}}$
Simplify.	$\frac{n}{(n-2)}$

TRY IT : : 7.39 Subtract : $\frac{3x-1}{x^2-5x-6} - \frac{2}{6-x}$.

TRY IT : : 7.40 Subtract: $\frac{-2y-2}{y^2+2y-8} - \frac{y-1}{2-y}$.

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

EXAMPLE 7.21

Subtract: $\frac{4}{a^2+6a+5} - \frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$
Factor the denominators.	$\frac{4}{(a+1)(a+5)}-\frac{3}{(a+2)(a+5)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{aligned} a^2+6a+5 &= (a+1)(a+5) \\ a^2+7a+10 &= (a+5)(a+2) \\ \hline \text{LCD} &= (a+1)(a+5)(a+2) \end{aligned}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{4(a+2)}{(a+1)(a+5)(a+2)}-\frac{3(a+1)}{(a+2)(a+5)(a+1)}$
Simplify the numerators.	$\frac{4a+8}{(a+1)(a+5)(a+2)}-\frac{3a+3}{(a+2)(a+5)(a+1)}$
Subtract the rational expressions.	$\frac{4a+8-(3a+3)}{(a+1)(a+5)(a+2)}$
Simplify the numerator.	$\frac{4a+8-3a+3}{(a+1)(a+5)(a+2)}$
	$\frac{a+5}{(a+1)(a+5)(a+2)}$
Look for common factors.	$\frac{\cancel{(a+5)}}{(a+1)\cancel{(a+5)}(a+2)}$
Simplify.	$\frac{1}{(a+1)(a+2)}$

TRY IT : : 7.41 Subtract: $\frac{3}{b^2-4b-5}-\frac{2}{b^2-6b+5}$.

TRY IT : : 7.42 Subtract: $\frac{4}{x^2-4}-\frac{3}{x^2-x-2}$.

We follow the same steps as before to find the LCD when we have more than two rational expressions. In the next example, we will start by factoring all three denominators to find their LCD.

EXAMPLE 7.22

Simplify: $\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$.

Solution

	$\frac{2u}{u-1}+\frac{1}{u}-\frac{2u-1}{u^2-u}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	

Find the LCD.	$u-1 = (u-1)$ $u = u$ $u^2-u = u(u-1)$ LCD $= u(u-1)$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{2u \cdot u}{(u-1)u} + \frac{1 \cdot (u-1)}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
	$\frac{2u^2}{(u-1)u} + \frac{u-1}{u \cdot (u-1)} - \frac{2u-1}{u(u-1)}$
Write as one rational expression.	$\frac{2u^2+u-1-2u+1}{u(u-1)}$
Simplify.	$\frac{2u^2-u}{u(u-1)}$
Factor the numerator, and remove common factors.	$\frac{\cancel{u}(2u-1)}{\cancel{u}(u-1)}$
Simplify.	$\frac{2u-1}{u-1}$

TRY IT : : 7.43 Simplify: $\frac{v}{v+1} + \frac{3}{v-1} - \frac{6}{v^2-1}$.

TRY IT : : 7.44 Simplify: $\frac{3w}{w+2} + \frac{2}{w+7} - \frac{17w+4}{w^2+9w+14}$.

Add and subtract rational functions

To add or subtract rational functions, we use the same techniques we used to add or subtract polynomial functions.

EXAMPLE 7.23

Find $R(x) = f(x) - g(x)$ where $f(x) = \frac{x+5}{x-2}$ and $g(x) = \frac{5x+18}{x^2-4}$.

Solution

	$R(x) = f(x) - g(x)$
Substitute in the functions $f(x)$, $g(x)$.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{x^2-4}$
Factor the denominators.	$R(x) = \frac{x+5}{x-2} - \frac{5x+18}{(x-2)(x+2)}$
Do the expressions have a common denominator? No. Rewrite each expression with the LCD.	
Find the LCD.	$x-2 = (x-2)$ $x^2-4 = (x-2)(x+2)$ LCD $= (x-2)(x+2)$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$R(x) = \frac{(x+5)(x+2)}{(x-2)(x+2)} - \frac{5x+18}{(x-2)(x+2)}$

Write as one rational expression.	$R(x) = \frac{(x+5)(x+2) - (5x+18)}{(x-2)(x+2)}$
Simplify.	$R(x) = \frac{x^2 + 7x + 10 - 5x - 18}{(x-2)(x+2)}$
	$R(x) = \frac{x^2 + 2x - 8}{(x-2)(x+2)}$
Factor the numerator, and remove common factors.	$R(x) = \frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$
Simplify.	$R(x) = \frac{(x+4)}{(x+2)}$

TRY IT : : 7.45 Find $R(x) = f(x) - g(x)$ where $f(x) = \frac{x+1}{x+3}$ and $g(x) = \frac{x+17}{x^2 - x - 12}$.

TRY IT : : 7.46 Find $R(x) = f(x) + g(x)$ where $f(x) = \frac{x-4}{x+3}$ and $g(x) = \frac{4x+6}{x^2 - 9}$.

MEDIA : :

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- **Add and Subtract Rational Expressions- Unlike Denominators (https://openstax.org/l/37AddSubRatExp)**

7.2 EXERCISES

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

75. $\frac{2}{15}+\frac{7}{15}$

76. $\frac{7}{24}+\frac{11}{24}$

77. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

78. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

79. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

80. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

81. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

82. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

In the following exercises, subtract.

83. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

84. $\frac{25b^2}{5b-6}-\frac{36}{5b-6}$

85. $\frac{3m^2}{6m-30}-\frac{21m-30}{6m-30}$

86. $\frac{2n^2}{4n-32}-\frac{18n-16}{4n-32}$

87. $\frac{6p^2+3p+4}{p^2+4p-5}-\frac{5p^2+p+7}{p^2+4p-5}$

88. $\frac{5q^2+3q-9}{q^2+6q+8}-\frac{4q^2+9q+7}{q^2+6q+8}$

89. $\frac{5r^2+7r-33}{r^2-49}-\frac{4r^2+5r+30}{r^2-49}$

90. $\frac{7t^2-t-4}{t^2-25}-\frac{6t^2+12t-44}{t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

91. $\frac{10v}{2v-1}+\frac{2v+4}{1-2v}$

92. $\frac{20w}{5w-2}+\frac{5w+6}{2-5w}$

93. $\frac{10x^2+16x-7}{8x-3}+\frac{2x^2+3x-1}{3-8x}$

94. $\frac{6y^2+2y-11}{3y-7}+\frac{3y^2-3y+17}{7-3y}$

95. $\frac{z^2+6z}{z^2-25}-\frac{3z+20}{25-z^2}$

96. $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

97. $\frac{2b^2+30b-13}{b^2-49}-\frac{2b^2-5b-8}{49-b^2}$

98. $\frac{c^2+5c-10}{c^2-16}-\frac{c^2-8c-10}{16-c^2}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, ⓐ find the LCD for the given rational expressions ⓑ rewrite them as equivalent rational expressions with the lowest common denominator.

99. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

100. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

101. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$

102. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

103. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

104. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-7c+10}$

105. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

106. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

107. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$

108. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

109. $\frac{3}{r+4}+\frac{2}{r-5}$

110. $\frac{4}{s-7}+\frac{5}{s+3}$

111. $\frac{5}{3w-2}+\frac{2}{w+1}$

112. $\frac{4}{2x+5}+\frac{2}{x-1}$

113. $\frac{2y}{y+3}+\frac{3}{y-1}$

114. $\frac{3z}{z-2}+\frac{1}{z+5}$

115. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

116. $\frac{4}{cd+3c}+\frac{1}{d^2-9}$

117. $\frac{-3m}{3m-3}+\frac{5m}{m^2+3m-4}$

118. $\frac{8}{4n+4}+\frac{6}{n^2-n-2}$

119. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

120. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

121. $\frac{t}{t-6}-\frac{t-2}{t+6}$

122. $\frac{x-3}{x+6}-\frac{x}{x+3}$

123. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

124. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

125. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

126. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

127. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

128. $\frac{7q+8}{q^2-2q-24}-\frac{q+2}{4-q}$

129. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

130. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

131. $\frac{2x+7}{10x-1}+3$

132. $\frac{8y-4}{5y+2}-6$

133. $\frac{3}{x^2-3x-4}-\frac{2}{x^2-5x+4}$

134. $\frac{4}{x^2-6x+5}-\frac{3}{x^2-7x+10}$

135. $\frac{5}{x^2+8x-9}-\frac{4}{x^2+10x+9}$

136. $\frac{3}{2x^2+5x+2}-\frac{1}{2x^2+3x+1}$

137. $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

138. $\frac{2b}{b-5}+\frac{3}{2b}-\frac{2b-15}{2b^2-10b}$

139. $\frac{c}{c+2}+\frac{5}{c-2}-\frac{11c}{c^2-4}$

140. $\frac{6d}{d-5}+\frac{1}{d+4}-\frac{7d-5}{d^2-d-20}$

141. $\frac{3d}{d+2}+\frac{4}{d}-\frac{d+8}{d^2+2d}$

142. $\frac{2q}{q+5}+\frac{3}{q-3}-\frac{13q+15}{q^2+2q-15}$

Add and Subtract Rational Functions

In the following exercises, find ⓐ $R(x) = f(x) + g(x)$ ⓑ $R(x) = f(x) - g(x)$.

143. $f(x) = \frac{-5x - 5}{x^2 + x - 6}$ and $g(x) = \frac{x + 1}{2 - x}$

144. $f(x) = \frac{-4x - 24}{x^2 + x - 30}$ and $g(x) = \frac{x + 7}{5 - x}$

145. $f(x) = \frac{6x}{x^2 - 64}$ and $g(x) = \frac{3}{x - 8}$

146. $f(x) = \frac{5}{x + 7}$ and $g(x) = \frac{10x}{x^2 - 49}$

Writing Exercises

147. Donald thinks that $\frac{3}{x} + \frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

148. Explain how you find the Least Common Denominator of $x^2 + 5x + 4$ and $x^2 - 16$.

149. Felipe thinks $\frac{1}{x} + \frac{1}{y}$ is $\frac{2}{x + y}$.

ⓐ Choose numerical values for *x* and *y* and evaluate $\frac{1}{x} + \frac{1}{y}$.

ⓑ Evaluate $\frac{2}{x + y}$ for the same values of *x* and *y* you used in part ⓐ.

ⓒ Explain why Felipe is wrong.

ⓓ Find the correct expression for $\frac{1}{x} + \frac{1}{y}$.

150. Simplify the expression $\frac{4}{n^2 + 6n + 9} - \frac{1}{n^2 - 9}$ and explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites.			
find the least common denominator of rational expressions.			
add and subtract rational expressions with unlike denominators.			
add or subtract rational functions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

7.3 Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.

   If you missed this problem, review Example 1.27.

2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.

   If you missed this problem, review Example 1.31.

3. Solve: $\frac{1}{2x}+\frac{1}{4}=\frac{1}{8}$.

   If you missed this problem, review Example 2.9.

Simplify a Complex Rational Expression by Writing it as Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \qquad \frac{\frac{x}{2}}{\frac{xy}{6}}$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Complex Rational Expression

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are a few complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}} \qquad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \qquad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter.

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}$$

We noted that fraction bars tell us to divide, so rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right)\div\left(\frac{2x^2-8x+3}{x^2-5x+6}\right).$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

EXAMPLE 7.24

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$.

Solution

$$\dfrac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$$

Rewrite the complex fraction as division.	$\dfrac{6}{x-4} \div \dfrac{3}{x^2-16}$
Rewrite as the product of fir t times the reciprocal of the second.	$\dfrac{6}{x-4} \cdot \dfrac{x^2-16}{3}$
Factor.	$\dfrac{3 \cdot 2}{x-4} \cdot \dfrac{(x-4)(x+4)}{3}$
Multiply.	$\dfrac{3 \cdot 2(x-4)(x+4)}{3(x-4)}$
Remove common factors.	$\dfrac{\cancel{3} \cdot 2\cancel{(x-4)}(x+4)}{\cancel{3}\cancel{(x-4)}}$
Simplify.	$2(x+4)$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

TRY IT : : 7.47

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{2}{x^2-1}}{\frac{3}{x+1}}$.

TRY IT : : 7.48

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}$.

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

EXAMPLE 7.25

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Simplify the numerator and denominator. Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.	$\dfrac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$\dfrac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$
Rewrite the complex rational expression as a division problem.	$\frac{3}{6}\div\frac{1}{6}$
Multiply the first by the reciprocal of the second.	$\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	3

TRY IT : : 7.49

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$.

TRY IT : : 7.50

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$.

We follow the same procedure when the complex rational expression contains variables.

EXAMPLE 7.26 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSION USING DIVISION

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Simplify the numerator and denominator.	We will simplify the sum in the numerator and difference in the denominator.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find a common denominator and add the fractions in the numerator.	$\dfrac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
		$\dfrac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	Find a common denominator and subtract the fractions in the denominator.	$\dfrac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
	We now have just one rational expression in the numerator and one in the denominator.	
Step 2. Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Step 3. Divide the expressions.	Multiply the first by the reciprocal of the second.	$\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$\frac{\cancel{xy}\cancel{(y+x)}}{\cancel{xy}(x-y)\cancel{(x+y)}}$
	Simplify.	$\frac{1}{x-y}$

TRY IT : : 7.51

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$.

TRY IT : : 7.52

Simplify the complex rational expression by writing it as division: $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}$.

We summarize the steps here.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY WRITING IT AS DIVISION.

Step 1. Simplify the numerator and denominator.
Step 2. Rewrite the complex rational expression as a division problem.
Step 3. Divide the expressions.

EXAMPLE 7.27

Simplify the complex rational expression by writing it as division: $\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$.

Solution

	$\dfrac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator. Find common denominators for the numerator and denominator.	$\dfrac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$\dfrac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$
Subtract the rational expressions in the numerator and add in the denominator.	$\dfrac{\frac{n^2+5n-4n}{n+5}}{\frac{n-5+n+5}{(n+5)(n-5)}}$
Simplify. (We now have one rational expression over one rational expression.)	$\dfrac{\frac{n^2+n}{n+5}}{\frac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$\frac{n^2+n}{n+5} \div \frac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$\frac{n^2+n}{n+5} \cdot \frac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$\frac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$\frac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$
Simplify.	$\frac{(n+1)(n-5)}{2}$

TRY IT : : 7.53

Simplify the complex rational expression by writing it as division: $\dfrac{b-\frac{3b}{b+5}}{\frac{2}{b+5}+\frac{1}{b-5}}$.

TRY IT : : 7.54

Simplify the complex rational expression by writing it as division: $\dfrac{1-\frac{3}{c+4}}{\frac{1}{c+4}+\frac{c}{3}}$.

Simplify a Complex Rational Expression by Using the LCD

We "cleared" the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.25. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

EXAMPLE 7.28

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$.

Solution

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6.	
Clear the fractions by multiplying the numerator and denominator by that LCD.	$\dfrac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$\dfrac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$\dfrac{2+1}{3-2}$
	$\dfrac{3}{1}$
	3

TRY IT : : 7.55

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}$.

TRY IT : : 7.56

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}$.

We will use the same example as in **Example 7.26**. Decide which method works better for you.

EXAMPLE 7.29 HOW TO SIMPLIFY A COMPLEX RATIONAL EXPRESSING USING THE LCD

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$.

Solution

Step 1. Find the LCD of all fractions in the complex rational expression.	The LCD of all the fractions is xy.	$\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Step 2. Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$\dfrac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$

Step 3. Simplify the expression.	Distribute.	$\dfrac{xy \cdot \frac{1}{x} + xy \cdot \frac{1}{y}}{xy \cdot \frac{x}{y} - xy \cdot \frac{y}{x}}$
		$\dfrac{y+x}{x^2-y^2}$
	Simplify.	$\dfrac{(y+x)}{(x-y)(x+y)}$
	Remove common factors.	$\dfrac{1}{x-y}$

TRY IT : : 7.57

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}$.

TRY IT : : 7.58

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}$.

HOW TO : : SIMPLIFY A COMPLEX RATIONAL EXPRESSION BY USING THE LCD.

Step 1. Find the LCD of all fractions in the complex rational expression.
Step 2. Multiply the numerator and denominator by the LCD.
Step 3. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

EXAMPLE 7.30

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$.

Solution

	$\dfrac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression. The LCD is $x^2-36=(x+6)(x-6)$.	
Multiply the numerator and denominator by the LCD.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Simplify the expression.	

Distribute in the denominator.	$\dfrac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$\dfrac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\frac{4}{\cancel{x-6}}\right)-\cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$\dfrac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$\dfrac{2(x-6)}{4x+20}$
Factor the denominator.	$\dfrac{2(x-6)}{4(x+5)}$
Remove common factors.	$\dfrac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$
Simplify.	$\dfrac{x-6}{2(x+5)}$
Notice that there are no more factors common to the numerator and denominator.	

TRY IT : : 7.59

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}$.

TRY IT : : 7.60

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}$.

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

EXAMPLE 7.31

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$.

Solution

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$

Find the LCD of all fractions in the complex rational expression.	
The LCD is $(m-3)(m-4)$.	
Multiply the numerator and denominator by the LCD.	$\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$\frac{\cancel{(m-3)}\cancel{(m-4)}\frac{4}{\cancel{(m-3)}\cancel{(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$
Simplify.	$\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$\frac{4}{3m-12-2m+6}$
Combine like terms.	$\frac{4}{m-6}$

TRY IT : : 7.61

Simplify the complex rational expression by using the LCD: $\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}$.

TRY IT : : 7.62

Simplify the complex rational expression by using the LCD: $\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}$.

EXAMPLE 7.32

Simplify the complex rational expression by using the LCD: $\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$.

Solution

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$

Find the LCD of all fractions in the complex rational expression.
The LCD is $(y+1)(y-1)$.

Multiply the numerator and denominator by the LCD.	$\dfrac{(y+1)(y-1)\dfrac{y}{y+1}}{(y+1)(y-1)\left(1+\dfrac{1}{y-1}\right)}$
Distribute in the denominator and simplify.	$\dfrac{\cancel{(y+1)}(y-1)\dfrac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\dfrac{1}{\cancel{y-1}}\right)}$
Simplify.	$\dfrac{(y-1)y}{(y+1)(y-1)+(y+1)}$
Simplify the denominator and leave the numerator factored.	$\dfrac{y(y-1)}{y^2-1+y+1}$
	$\dfrac{y(y-1)}{y^2+y}$
Factor the denominator and remove factors common with the numerator.	$\dfrac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$
Simplify.	$\dfrac{y-1}{y+1}$

TRY IT : : 7.63

Simplify the complex rational expression by using the LCD: $\dfrac{\frac{x}{x+3}}{1+\frac{1}{x+3}}$.

TRY IT : : 7.64

Simplify the complex rational expression by using the LCD: $\dfrac{1+\frac{1}{x-1}}{\frac{3}{x+1}}$.

MEDIA : :

Access this online resource for additional instruction and practice with complex fractions.

- Complex Fractions (https://openstax.org/l/37CompFrac)

7.3 EXERCISES

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

151. $\dfrac{\frac{2a}{a+4}}{\frac{4a^2}{a^2-16}}$

152. $\dfrac{\frac{3b}{b-5}}{\frac{b^2}{b^2-25}}$

153. $\dfrac{\frac{5}{c^2+5c-14}}{\frac{10}{c+7}}$

154. $\dfrac{\frac{8}{d^2+9d+18}}{\frac{12}{d+6}}$

155. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

156. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

157. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

158. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

159. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

160. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

161. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

162. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

163. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

164. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}+\frac{2}{y+4}}$

165. $\dfrac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$

166. $\dfrac{4+\frac{4}{b-5}}{\frac{1}{b-5}+\frac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

167. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

168. $\dfrac{\frac{1}{4}+\frac{1}{9}}{\frac{1}{6}+\frac{1}{12}}$

169. $\dfrac{\frac{5}{6}+\frac{2}{9}}{\frac{7}{18}-\frac{1}{3}}$

170. $\dfrac{\frac{1}{6}+\frac{4}{15}}{\frac{3}{5}-\frac{1}{2}}$

171. $\dfrac{\frac{c}{d}+\frac{1}{d}}{\frac{1}{d}-\frac{d}{c}}$

172. $\dfrac{\frac{1}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

173. $\dfrac{\frac{1}{p}+\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$

174. $\dfrac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

175. $\dfrac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$

176. $\dfrac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$

177. $\dfrac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$

178. $\dfrac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$

179. $\dfrac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$

180. $\dfrac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$

181. $\dfrac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$

182. $\dfrac{\frac{6}{d-4}-\frac{2}{d+7}}{\frac{2d}{d^2+3d-28}}$

183. $\dfrac{2+\frac{1}{p-3}}{\frac{5}{p-3}}$

184. $\dfrac{\frac{n}{n-2}}{3+\frac{5}{n-2}}$

185. $\dfrac{\frac{m}{m+5}}{4+\frac{1}{m-5}}$

186. $\dfrac{7+\frac{2}{q-2}}{\frac{1}{q+2}}$

In the following exercises, simplify each complex rational expression using either method.

187. $\dfrac{\frac{3}{4}-\frac{2}{7}}{\frac{1}{2}+\frac{5}{14}}$

188. $\dfrac{\frac{v}{w}+\frac{1}{v}}{\frac{1}{v}-\frac{v}{w}}$

189. $\dfrac{\frac{2}{a+4}}{\frac{1}{a^2-16}}$

190. $\dfrac{\frac{3}{b^2-3b-40}}{\frac{5}{b+5}-\frac{2}{b-8}}$

191. $\dfrac{\frac{3}{m}+\frac{3}{n}}{\frac{1}{m^2}-\frac{1}{n^2}}$

192. $\dfrac{\frac{2}{r-9}}{\frac{1}{r+9}+\frac{3}{r^2-81}}$

193. $\dfrac{x-\frac{3x}{x+2}}{\frac{3}{x+2}+\frac{3}{x-2}}$

194. $\dfrac{\frac{y}{y+3}}{2+\frac{1}{y-3}}$

Writing Exercises

195. In this section, you learned to simplify the complex fraction $\dfrac{\frac{3}{x+2}}{\frac{x}{x^2-4}}$ two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

196. Efraim wants to start simplifying the complex fraction $\dfrac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$ by cancelling the variables from the numerator and denominator, $\dfrac{\frac{1}{\cancel{a}}+\frac{1}{\cancel{b}}}{\frac{1}{\cancel{a}}-\frac{1}{\cancel{b}}}$. Explain what is wrong with Efraim's plan.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify a complex rational expression by writing it as division.			
simplify a complex rational expression by using the LCD.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

7.4 Solve Rational Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Be Prepared!

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x + \frac{1}{2} = \frac{1}{3}$.
   If you missed this problem, review **Example 2.9**.
2. Solve: $n^2 - 5n - 36 = 0$.
   If you missed this problem, review **Example 6.45**.
3. Solve the formula $5x + 2y = 10$ for y.
   If you missed this problem, review **Example 2.31**.

After defining the terms 'expression' and 'equation' earlier, we have used them throughout this book. We have *simplified* many kinds of *expressions* and *solved* many kinds of *equations*. We have simplified many rational expressions so far in this chapter. Now we will *solve* a **rational equation**.

Rational Equation

A **rational equation** is an equation that contains a rational expression.

You must make sure to know the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x + \frac{1}{2}$	$\frac{1}{8}x + \frac{1}{2} = \frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36} = y+1$
$\frac{1}{n-3} + \frac{1}{n+4}$	$\frac{1}{n-3} + \frac{1}{n+4} = \frac{15}{n^2+n-12}$

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to "clear" the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Extraneous Solution to a Rational Equation

An **extraneous solution to a rational equation** is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, *c*, by writing $x \neq c$ next to the equation.

EXAMPLE 7.33 HOW TO SOLVE A RATIONAL EQUATION

Solve: $\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$.

Solution

Step 1. Note any value of the variable that would make any denominator zero.	If $x = 0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}, x \neq 0$
Step 2. Find the least common denominator of *all* denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$.	The LCD is $6x$.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. Use the Distributive Property. Simplify – and notice, no more fractions!	$6x \cdot \left(\frac{1}{x} + \frac{1}{3}\right) = 6x \cdot \left(\frac{5}{6}\right)$ $6x \cdot \frac{1}{x} + 6x \cdot \frac{1}{3} = 6x \cdot \left(\frac{5}{6}\right)$ $6 + 2x = 5x$
Step 4. Solve the resulting equation.	Simplify.	$6 = 3x$ $2 = x$
Step 5. Check. • If any values found in Step 1 are algebraic solutions, discard them. • Check any remaining solutions in the original equation.	We did not get 0 as an algebraic solution. We substitute $x = 2$ into the original equation.	$\frac{1}{x} + \frac{1}{3} = \frac{5}{6}$ $\frac{1}{2} + \frac{1}{3} \stackrel{?}{=} \frac{5}{6}$ $\frac{3}{6} + \frac{2}{6} \stackrel{?}{=} \frac{5}{6}$ $\frac{5}{6} = \frac{5}{6}$ ✓ The solution is $x = 2$.

TRY IT : : 7.65 Solve: $\frac{1}{y} + \frac{2}{3} = \frac{1}{5}$.

TRY IT : : 7.66 Solve: $\frac{2}{3} + \frac{1}{5} = \frac{1}{x}$.

The steps of this method are shown.

HOW TO : : SOLVE EQUATIONS WITH RATIONAL EXPRESSIONS.

Step 1. Note any value of the variable that would make any denominator zero.
Step 2. Find the least common denominator of *all* denominators in the equation.
Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
Step 4. Solve the resulting equation.
Step 5. Check:
- If any values found in Step 1 are algebraic solutions, discard them.
- Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

EXAMPLE 7.34 HOW TO SOLVE A RATIONAL EQUATION USING THE ZERO PRODUCT PROPERTY

Solve: $1 - \frac{5}{y} = -\frac{6}{y^2}$.

Solution

	$1 - \frac{5}{y} = -\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1 - \frac{5}{y} = -\frac{6}{y^2}, y \neq 0$
Find the least common denominator of all denominators in the equation. The LCD is y^2.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1 - \frac{5}{y}\right) = y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2 \cdot 1 - y^2\left(\frac{5}{y}\right) = y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2 - 5y = -6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2 - 5y + 6 = 0$
Factor.	$(y - 2)(y - 3) = 0$
Use the Zero Product Property.	$y - 2 = 0$ or $y - 3 = 0$
Solve.	$y = 2$ or $y = 3$

Check.
We did not get 0 as an algebraic solution.

Check $y = 2$ and $y = 3$ in the original equation.

$1 - \frac{5}{y} = -\frac{6}{y^2}$	$1 - \frac{5}{y} = -\frac{6}{y^2}$
$1 - \frac{5}{2} \overset{?}{=} -\frac{6}{2^2}$	$1 - \frac{5}{3} \overset{?}{=} -\frac{6}{3^2}$
$1 - \frac{5}{2} \overset{?}{=} -\frac{6}{4}$	$1 - \frac{5}{3} \overset{?}{=} -\frac{6}{9}$
$\frac{2}{2} - \frac{5}{2} \overset{?}{=} -\frac{6}{4}$	$\frac{3}{3} - \frac{5}{3} \overset{?}{=} -\frac{6}{9}$
$-\frac{3}{2} \overset{?}{=} -\frac{6}{4}$	$-\frac{2}{3} \overset{?}{=} -\frac{6}{9}$
$-\frac{3}{2} = -\frac{3}{2}$ ✓	$-\frac{2}{3} = -\frac{2}{3}$ ✓

The solution is $y = 2, \quad y = 3.$

TRY IT : : 7.67 Solve: $1 - \frac{2}{x} = \frac{15}{x^2}.$

TRY IT : : 7.68 Solve: $1 - \frac{4}{y} = \frac{12}{y^2}.$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

EXAMPLE 7.35

Solve: $\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}.$

Solution

	$\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{x^2-4}$
Note any value of the variable that would make any denominator zero.	$\frac{2}{x+2} + \frac{4}{x-2} = \frac{x-1}{(x+2)(x-2)}, x \neq -2, x \neq 2$
Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(x+2)(x-2)\left(\frac{2}{x+2} + \frac{4}{x-2}\right) = (x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$
Distribute.	$(x+2)(x-2)\frac{2}{x+2} + (x+2)(x-2)\frac{4}{x-2} = (x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$

Remove common factors.	$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}} + (x+2)\cancel{(x-2)}\frac{4}{\cancel{x-2}} = \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$
Simplify.	$2(x-2)+4(x+2)=x-1$
Distribute.	$2x-4+4x+8=x-1$
Solve.	$6x+4=x-1$ $5x=-5$ $x=-1$

Check:
We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

$$\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$$

$$\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$$

$$\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$$

$$\frac{2}{3}=\frac{2}{3}\checkmark$$

The solution is $x=-1$.

TRY IT : : 7.69 Solve: $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.

TRY IT : : 7.70 Solve: $\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$.

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

EXAMPLE 7.36

Solve: $\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$.

Solution

	$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}=\frac{3}{m-1}$
Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.	$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$
Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$.	
Clear the fractions by multiplying both sides of the equation by the LCD.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$
Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$
Remove common factors.	$\cancel{(m-4)}\cancel{(m-1)}\left(\frac{m+11}{\cancel{(m-4)}\cancel{(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\cancel{(m-1)}\frac{3}{\cancel{m-1}}$
Simplify.	$m+11=5(m-1)-3(m-4)$
Solve the resulting equation.	$m+11=5m-5-3m+12$ $4=m$
Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.	
	There is no solution to this equation.

TRY IT : : 7.71 Solve: $\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$.

TRY IT : : 7.72 Solve: $\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

EXAMPLE 7.37

Solve: $\frac{y}{y+6}=\frac{72}{y^2-36}+4$.

Solution

	$\dfrac{y}{y+6} = \dfrac{72}{y^2-36} + 4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\dfrac{y}{y+6} = \dfrac{72}{(y-6)(y+6)} + 4, y \neq 6, y \neq -6$
Find the least common denominator. The LCD is $(y-6)(y+6)$.	
Clear the fractions.	$(y-6)(y+6)\left(\dfrac{y}{y+6}\right) = (y-6)(y+6)\left(\dfrac{72}{(y-6)(y+6)} + 4\right)$
Simplify.	$(y-6) \cdot y = 72 + (y-6)(y+6) \cdot 4$
Simplify.	$y(y-6) = 72 + 4(y^2-36)$
Solve the resulting equation.	$y^2 - 6y = 72 + 4y^2 - 144$ $0 = 3y^2 + 6y - 72$ $0 = 3(y^2 + 2y - 24)$ $0 = 3(y+6)(y-4)$ $y = -6, y = 4$
Check. $y = -6$ is an extraneous solution. Check $y = 4$ in the original equation. $\dfrac{y}{y+6} = \dfrac{72}{y^2-36} + 4$ $\dfrac{4}{4+6} \stackrel{?}{=} \dfrac{72}{4^2-36} + 4$ $\dfrac{4}{10} \stackrel{?}{=} \dfrac{72}{-20} + 4$ $\dfrac{4}{10} \stackrel{?}{=} -\dfrac{36}{10} + \dfrac{40}{10}$ $\dfrac{4}{10} = \dfrac{4}{10}$ ✓	
	The solution is $y = 4$.

TRY IT : : 7.73 Solve: $\dfrac{x}{x+4} = \dfrac{32}{x^2-16} + 5.$

TRY IT : : 7.74 Solve: $\dfrac{y}{y+8} = \dfrac{128}{y^2-64} + 9.$

In some cases, all the algebraic solutions are extraneous.

EXAMPLE 7.38

Solve: $\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$.

Solution

	$\frac{x}{2x-2} - \frac{2}{3x+3} = \frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)} - \frac{2}{3(x+1)} = \frac{5x^2-2x+9}{12(x-1)(x+1)}, x \neq 1, x \neq -1$
Find the least common denominator. The LCD is $12(x-1)(x+1)$.	
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)} - \frac{2}{3(x+1)}\right) = 12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1) \cdot x - 4(x-1) \cdot 2 = 5x^2-2x+9$
Simplify.	$6x(x+1) - 4 \cdot 2(x-1) = 5x^2-2x+9$
Solve the resulting equation.	$6x^2+6x-8x+8 = 5x^2-2x+9$ $x^2-1=0$ $(x-1)(x+1)=0$ $x=1$ or $x=-1$
Check. $x=1$ and $x=-1$ are extraneous solutions.	
	The equation has no solution.

TRY IT : : 7.75

Solve: $\frac{y}{5y-10} - \frac{5}{3y+6} = \frac{2y^2-19y+54}{15y^2-60}$.

TRY IT : : 7.76

Solve: $\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-16}{8z^2+2z-64}$.

EXAMPLE 7.39

Solve: $\frac{4}{3x^2-10x+3} + \frac{3}{3x^2+2x-1} = \frac{2}{x^2-2x-3}$.

Solution

	$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}$ $x\neq-1,\ x\neq\frac{1}{3},\ x\neq3$
Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.	
Clear the fractions.	
	$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$
Simplify.	$4(x+1)+3(x-3)=2(3x-1)$
Distribute.	$4x+4+3x-9=6x-2$
Simplify.	$7x-5=6x-2$
	$x=3$
The only algebraic solution was $x=3,$ but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.	
	There is no solution to this equation.

TRY IT : : 7.77 Solve: $\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}.$

TRY IT : : 7.78 Solve: $\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}.$

Use Rational Functions

Working with functions that are defined by rational expressions often lead to rational equations. Again, we use the same techniques to solve them.

EXAMPLE 7.40

For rational function, $f(x)=\frac{2x-6}{x^2-8x+15},$ ⓐ find the domain of the function, ⓑ solve $f(x)=1,$ and ⓒ find the points on the graph at this function value.

Solution

ⓐ The domain of a rational function is all real numbers except those that make the rational expression undefined. So to find them, we will set the denominator equal to zero and solve.

	$x^2 - 8x + 15 = 0$
Factor the trinomial.	$(x-3)(x-5) = 0$
Use the Zero Product Property.	$x - 3 = 0 \quad x - 5 = 0$
Solve.	$x = 3 \quad x = 5$

The domain is all real numbers except $x \neq 3,\ x \neq 5$.

ⓑ

	$f(x) = 1$
Substitute in the rational expression.	$\frac{2x-6}{x^2-8x+15} = 1$
Factor the denominator.	$\frac{2x-6}{(x-3)(x-5)} = 1$
Multiply both sides by the LCD, $(x-3)(x-5)$.	$(x-3)(x-5)\left(\frac{2x-6}{(x-3)(x-5)}\right) = (x-3)(x-5)(1)$
Simplify.	$2x - 6 = x^2 - 8x + 15$
Solve.	$0 = x^2 - 10x + 21$
Factor.	$0 = (x-7)(x-3)$
Use the Zero Product Property.	$x - 7 = 0 \quad x - 3 = 0$
Solve.	$x = 7 \quad x = 3$

ⓒ The value of the function is 1 when $x = 7,\ x = 3$. So the points on the graph of this function when $f(x) = 1$, will be $(7,\ 1),\ (3,\ 1)$.

TRY IT : : 7.79

For rational function, $f(x) = \frac{8-x}{x^2-7x+12}$, ⓐ find the domain of the function ⓑ solve $f(x) = 3$ ⓒ find the points on the graph at this function value.

TRY IT : : 7.80

For rational function, $f(x) = \frac{x-1}{x^2-6x+5}$, ⓐ find the domain of the function ⓑ solve $f(x) = 4$ ⓒ find the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

	$m = \frac{y - y_1}{x - x_1}$
Multiply both sides of the equation by $x - x_1$.	$m(x - x_1) = \left(\frac{y - y_1}{x - x_1}\right)(x - x_1)$
Simplify.	$m(x - x_1) = y - y_1$
Rewrite the equation with the y terms on the left.	$y - y_1 = m(x - x_1)$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,\ 3)$. We will add one more step to solve for *y*.

EXAMPLE 7.41

Solve: $m = \frac{y-2}{x-3}$ for y.

Solution

	$m = \frac{y-2}{x-3}$
Note any value of the variable that would make any denominator zero.	$m = \frac{y-2}{x-3}, x \neq 3$
Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.	$(x-3)m = (x-3)\left(\frac{y-2}{x-3}\right)$
Simplify.	$xm - 3m = y - 2$
Isolate the term with y.	$xm - 3m + 2 = y$

TRY IT : : 7.81 Solve: $m = \frac{y-5}{x-4}$ for y.

TRY IT : : 7.82 Solve: $m = \frac{y-1}{x+5}$ for y.

Remember to multiply both sides by the LCD in the next example.

EXAMPLE 7.42

Solve: $\frac{1}{c} + \frac{1}{m} = 1$ for c.

Solution

	$\frac{1}{c} + \frac{1}{m} = 1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c} + \frac{1}{m} = 1, c \neq 0, m \neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, cm.	$cm\left(\frac{1}{c} + \frac{1}{m}\right) = cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right) + cm\frac{1}{m} = cm(1)$
Simplify.	$m + c = cm$
Collect the terms with c to the right.	$m = cm - c$
Factor the expression on the right.	$m = c(m-1)$
To isolate c, divide both sides by $m-1$.	$\frac{m}{m-1} = \frac{c(m-1)}{m-1}$
Simplify by removing common factors.	$\frac{m}{m-1} = c$

Notice that even though we excluded $c = 0$, $m = 0$ from the original equation, we must also now state that $m \neq 1$.

TRY IT : : 7.83 Solve: $\frac{1}{a}+\frac{1}{b}=c$ for *a*.

TRY IT : : 7.84 Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for *y*.

MEDIA : :

Access this online resource for additional instruction and practice with equations with rational expressions.

- **Equations with Rational Expressions (https://openstax.org/l/37EqRatExp)**

7.4 EXERCISES

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve each rational equation.

197. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

198. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

199. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

200. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

201. $1-\frac{2}{m}=\frac{8}{m^2}$

202. $1+\frac{4}{n}=\frac{21}{n^2}$

203. $1+\frac{9}{p}=\frac{-20}{p^2}$

204. $1-\frac{7}{q}=\frac{-6}{q^2}$

205. $\frac{5}{3v-2}=\frac{7}{4v}$

206. $\frac{8}{2w+1}=\frac{3}{w}$

207. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

208. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

209. $\frac{8}{z-10}-\frac{7}{z+10}=\frac{5}{z^2-100}$

210. $\frac{9}{a+11}-\frac{6}{a-11}=\frac{6}{a^2-121}$

211. $\frac{-10}{q-2}-\frac{7}{q+4}=1$

212. $\frac{2}{s+7}-\frac{3}{s-3}=1$

213. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

214. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

215. $\frac{x-10}{x^2+8x+12}=\frac{3}{x+2}+\frac{4}{x+6}$

216. $\frac{y-5}{y^2-4y-5}=\frac{1}{y+1}+\frac{1}{y-5}$

217. $\frac{b+3}{3b}+\frac{b}{24}=\frac{1}{b}$

218. $\frac{c+3}{12c}+\frac{c}{36}=\frac{1}{4c}$

219. $\frac{d}{d+3}=\frac{18}{d^2-9}+4$

220. $\frac{m}{m+5}=\frac{50}{m^2-25}+6$

221. $\frac{n}{n+2}-3=\frac{8}{n^2-4}$

222. $\frac{p}{p+7}-8=\frac{98}{p^2-49}$

223. $\frac{q}{3q-9}-\frac{3}{4q+12}=\frac{7q^2+6q+63}{24q^2-216}$

224. $\frac{r}{3r-15}-\frac{1}{4r+20}=\frac{3r^2+17r+40}{12r^2-300}$

225. $\frac{s}{2s+6}-\frac{2}{5s+5}=\frac{5s^2-3s-7}{10s^2+40s+30}$

226. $\frac{t}{6t-12}-\frac{5}{2t+10}=\frac{t^2-23t+70}{12t^2+36t-120}$

227. $\frac{2}{x^2+2x-8} - \frac{1}{x^2+9x+20} = \frac{4}{x^2+3x-10}$

228. $\frac{5}{x^2+4x+3} + \frac{2}{x^2+x-6} = \frac{3}{x^2-x-2}$

229. $\frac{3}{x^2-5x-6} + \frac{3}{x^2-7x+6} = \frac{6}{x^2-1}$

230. $\frac{2}{x^2+2x-3} + \frac{3}{x^2+4x+3} = \frac{6}{x^2-1}$

Solve Rational Equations that Involve Functions

231. For rational function, $f(x) = \frac{x-2}{x^2+6x+8}$,

ⓐ find the domain of the function

ⓑ solve $f(x) = 5$

ⓒ find the points on the graph at this function value.

232. For rational function, $f(x) = \frac{x+1}{x^2-2x-3}$,

ⓐ find the domain of the function

ⓑ solve $f(x) = 1$

ⓒ find the points on the graph at this function value.

233. For rational function, $f(x) = \frac{2-x}{x^2-7x+10}$,

ⓐ find the domain of the function

ⓑ solve $f(x) = 2$

ⓒ find the points on the graph at this function value.

234. For rational function, $f(x) = \frac{5-x}{x^2+5x+6}$,

ⓐ find the domain of the function

ⓑ solve $f(x) = 3$

ⓒ the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve.

235. $\frac{C}{r} = 2\pi$ for r.

236. $\frac{I}{r} = P$ for r.

237. $\frac{v+3}{w-1} = \frac{1}{2}$ for w.

238. $\frac{x+5}{2-y} = \frac{4}{3}$ for y.

239. $a = \frac{b+3}{c-2}$ for c.

240. $m = \frac{n}{2-n}$ for n.

241. $\frac{1}{p} + \frac{2}{q} = 4$ for p.

242. $\frac{3}{s} + \frac{1}{t} = 2$ for s.

243. $\frac{2}{v} + \frac{1}{5} = \frac{3}{w}$ for w.

244. $\frac{6}{x} + \frac{2}{3} = \frac{1}{y}$ for y.

245. $\frac{m+3}{n-2} = \frac{4}{5}$ for n.

246. $r = \frac{s}{3-t}$ for t.

247. $\frac{E}{c} = m^2$ for c.

248. $\frac{R}{T} = W$ for T.

249. $\frac{3}{x} - \frac{5}{y} = \frac{1}{4}$ for y.

250. $c = \frac{2}{a} + \frac{b}{5}$ for a.

Writing Exercises

251. Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

252. Alek thinks the equation $\frac{y}{y+6} = \frac{72}{y^2-36} + 4$ has two solutions, $y = -6$ and $y = 4$. Explain why Alek is wrong.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve rational equations.			
solve rational equations involving functions.			
solve rational equations for a specific variable.			

ⓑ *On a scale of* $1-10,$ *how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

CHAPTER 7 REVIEW

KEY TERMS

complex rational expression A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

critical point of a rational inequality The critical point of a rational inequality is a number which makes the rational expression zero or undefined.

extraneous solution to a rational equation An extraneous solution to a rational equation is an algebraic solution that would cause any of the expressions in the original equation to be undefined.

proportion When two rational expressions are equal, the equation relating them is called a proportion.

rational equation A rational equation is an equation that contains a rational expression.

rational expression A rational expression is an expression of the form $\frac{p}{q}$, where *p* and *q* are polynomials and $q \neq 0$.

rational function A rational function is a function of the form $R(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)$ is not zero.

rational inequality A rational inequality is an inequality that contains a rational expression.

similar figures Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

simplified rational expression A simplified rational expression has no common factors, other than 1, in its numerator and denominator.

KEY CONCEPTS

7.1 Multiply and Divide Rational Expressions

- **Determine the values for which a rational expression is undefined.**
  Step 1. Set the denominator equal to zero.
  Step 2. Solve the equation.
- **Equivalent Fractions Property**
  If *a*, *b*, and *c* are numbers where $b \neq 0, c \neq 0,$ then $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ and $\frac{a \cdot c}{b \cdot c} = \frac{a}{b}$.
- **How to simplify a rational expression.**
  Step 1. Factor the numerator and denominator completely.
  Step 2. Simplify by dividing out common factors.
- **Opposites in a Rational Expression**
  The opposite of $a - b$ is $b - a$.
  $$\frac{a-b}{b-a} = -1 \qquad a \neq b$$
  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0, s \neq 0,$ then
  $$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$
- **How to multiply rational expressions.**
  Step 1. Factor each numerator and denominator completely.
  Step 2. Multiply the numerators and denominators.
  Step 3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  If *p*, *q*, *r*, and *s* are polynomials where $q \neq 0, r \neq 0, s \neq 0,$ then
  $$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

- **How to divide rational expressions.**
  Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  Step 2. Factor the numerators and denominators completely.
  Step 3. Multiply the numerators and denominators together.
  Step 4. Simplify by dividing out common factors.
- **How to determine the domain of a rational function.**
  Step 1. Set the denominator equal to zero.
  Step 2. Solve the equation.
  Step 3. The domain is all real numbers excluding the values found in Step 2.

7.2 Add and Subtract Rational Expressions

- **Rational Expression Addition and Subtraction**
  If *p*, *q*, and *r* are polynomials where $r \neq 0,$ then
  $$\frac{p}{r} + \frac{q}{r} = \frac{p+q}{r} \text{ and } \frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}$$
- **How to find the least common denominator of rational expressions.**
  Step 1. Factor each expression completely.
  Step 2. List the factors of each expression. Match factors vertically when possible.
  Step 3. Bring down the columns.
  Step 4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  Step 1. Determine if the expressions have a common denominator.
    - Yes – go to step 2.
    - No – Rewrite each rational expression with the LCD.
      - Find the LCD.
      - Rewrite each rational expression as an equivalent rational expression with the LCD.

  Step 2. Add or subtract the rational expressions.
  Step 3. Simplify, if possible.

7.3 Simplify Complex Rational Expressions

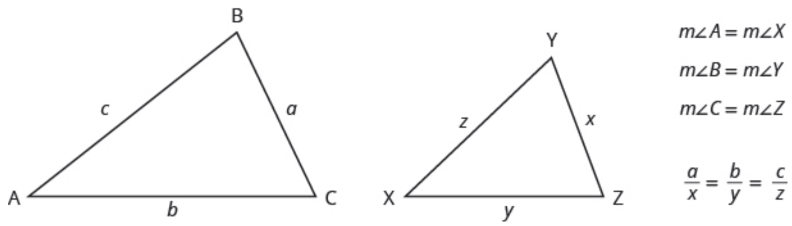
- **How to simplify a complex rational expression by writing it as division.**
  Step 1. Simplify the numerator and denominator.
  Step 2. Rewrite the complex rational expression as a division problem.
  Step 3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  Step 1. Find the LCD of all fractions in the complex rational expression.
  Step 2. Multiply the numerator and denominator by the LCD.
  Step 3. Simplify the expression.

7.4 Solve Rational Equations

- **How to solve equations with rational expressions.**
  Step 1. Note any value of the variable that would make any denominator zero.
  Step 2. Find the least common denominator of all denominators in the equation.
  Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
  Step 4. Solve the resulting equation.
  Step 5. Check:
    - If any values found in Step 1 are algebraic solutions, discard them.
    - Check any remaining solutions in the original equation.

7.5 Solve Applications with Rational Equations

- A proportion is an equation of the form $\frac{a}{b} = \frac{c}{d}$, where $b \neq 0, d \neq 0$. The proportion is read "*a* is to *b* as *c* is to *d*."
- **Property of Similar Triangles**
  If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



- **Direct Variation**
  - For any two variables *x* and *y*, *y* varies directly with *x* if $y = kx$, where $k \neq 0$. The constant *k* is called the constant of variation.
  - How to solve direct variation problems.
    Step 1. Write the formula for direct variation.
    Step 2. Substitute the given values for the variables.
    Step 3. Solve for the constant of variation.
    Step 4. Write the equation that relates x and y.
- **Inverse Variation**
  - For any two variables *x* and *y*, *y* varies inversely with *x* if $y = \frac{k}{x}$, where $k \neq 0$. The constant *k* is called the constant of variation.
  - How to solve inverse variation problems.
    Step 1. Write the formula for inverse variation.
    Step 2. Substitute the given values for the variables.
    Step 3. Solve for the constant of variation.
    Step 4. Write the equation that relates *x* and *y*.

7.6 Solve Rational Inequalities

- **Solve a rational inequality.**
  Step 1. Write the inequality as one quotient on the left and zero on the right.
  Step 2. Determine the critical points–the points where the rational expression will be zero or undefined.
  Step 3. Use the critical points to divide the number line into intervals.
  Step 4. Test a value in each interval. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.
  Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

REVIEW EXERCISES

7.1 Simplify, Multiply, and Divide Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

377. $\frac{5a+3}{3a-2}$

378. $\frac{b-7}{b^2-25}$

379. $\frac{5x^2y^2}{8y}$

380. $\frac{x-3}{x^2-x-30}$

Simplify Rational Expressions

In the following exercises, simplify.

381. $\frac{18}{24}$

382. $\frac{9m^4}{18mn^3}$

383. $\frac{x^2+7x+12}{x^2+8x+16}$

384. $\frac{7v-35}{25-v^2}$

Multiply Rational Expressions

In the following exercises, multiply.

385. $\frac{5}{8}\cdot\frac{4}{15}$

386. $\frac{3xy^2}{8y^3}\cdot\frac{16y^2}{24x}$

387. $\frac{72x-12x^2}{8x+32}\cdot\frac{x^2+10x+24}{x^2-36}$

388. $\frac{6y^2-2y-10}{9-y^2}\cdot\frac{y^2-6y+9}{6y^2+29y-20}$

Divide Rational Expressions

In the following exercises, divide.

389. $\frac{x^2-4x+12}{x^2+8x+12}\div\frac{x^2-36}{3x}$

390. $\frac{y^2-16}{4}\div\frac{y^3-64}{2y^2+8y+32}$

391. $\frac{11+w}{w-9}\div\frac{121-w^2}{9-w}$

392. $\frac{3y^2-12y-63}{4y+3}\div(6y^2-42y)$

393. $\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$

394. $\frac{8a^2+16a}{a-4}\cdot\frac{a^2+2a-24}{a^2+7a+10}\div\frac{2a^2-6a}{a+5}$

Multiply and Divide Rational Functions

395. Find $R(x)=f(x)\cdot g(x)$ where $f(x)=\frac{9x^2+9x}{x^2-3x-4}$ and $g(x)=\frac{x^2-16}{3x^2+12x}$.

396. Find $R(x)=\frac{f(x)}{g(x)}$ where $f(x)=\frac{27x^2}{3x-21}$ and $g(x)=\frac{9x^2+54x}{x^2-x-42}$.

7.2 Add and Subtract Rational Expressions

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, perform the indicated operations.

397. $\frac{7}{15}+\frac{8}{15}$

398. $\frac{4a^2}{2a-1}-\frac{1}{2a-1}$

399. $\frac{y^2+10y}{y+5}+\frac{25}{y+5}$

400. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

401. $\frac{x^2}{x-7}-\frac{3x+28}{x-7}$

402. $\frac{y^2}{y+11}-\frac{121}{y+11}$

403. $\frac{4q^2-q+3}{q^2+6q+5}-\frac{3q^2-q-6}{q^2+6q+5}$

404. $\frac{5t+4t+3}{t^2-25}-\frac{4t^2-8t-32}{t^2-25}$

Add and Subtract Rational Expressions Whose Denominators Are Opposites

In the following exercises, add and subtract.

405. $\frac{18w}{6w-1}+\frac{3w-2}{1-6w}$

406. $\frac{a^2+3a}{a^2-4}-\frac{3a-8}{4-a^2}$

407. $\frac{2b^2+3b-15}{b^2-49}-\frac{b^2+16b-1}{49-b^2}$

408. $\frac{8y^2-10y+7}{2y-5}+\frac{2y^2+7y+2}{5-2y}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

409. $\frac{7}{a^2-3a-10}, \frac{3a}{a^2-a-20}$

410. $\frac{6}{n^2-4}, \frac{2n}{n^2-4n+4}$

411. $\frac{5}{3p^2+17p-6}, \frac{2m}{3p^2-23p-8}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

412. $\frac{7}{5a}+\frac{3}{2b}$

413. $\frac{2}{c-2}+\frac{9}{c+3}$

414. $\frac{3x}{x^2-9}+\frac{5}{x^2+6x+9}$

415. $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

416. $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

417. $\frac{3y}{y+2}-\frac{y+2}{y+8}$

418. $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

419. $\frac{7m+3}{m+2}-5$

420. $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

421. $\frac{8a}{a^2-64}-\frac{4}{a+8}$

422. $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

Add and Subtract Rational Functions

In the following exercises, find $R(x)=f(x)+g(x)$ where $f(x)$ and $g(x)$ are given.

423. $f(x)=\frac{2x^2+12x-11}{x^2+3x-10}, \quad g(x)=\frac{x+1}{2-x}$

424. $f(x)=\frac{-4x+31}{x^2+x-30}, \quad g(x)=\frac{5}{x+6}$

In the following exercises, find $R(x)=f(x)-g(x)$ where $f(x)$ and $g(x)$ are given.

425. $f(x)=\frac{4x}{x^2-121}, \quad g(x)=\frac{2}{x-11}$

426. $f(x)=\frac{7}{x+6}, \quad g(x)=\frac{14x}{x^2-36}$

7.3 Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

427. $\dfrac{\frac{7x}{x+2}}{\frac{14x^2}{x^2-4}}$

428. $\dfrac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

429. $\dfrac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

430. $\dfrac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

431. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

432. $\dfrac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

433. $\dfrac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

434. $\dfrac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

7.4 7.4 Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

435. $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

436. $1-\frac{2}{m}=\frac{8}{m^2}$

437. $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

438. $\frac{3}{q+8}-\frac{2}{q-2}=1$

439. $\frac{v-15}{v^2-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

440. $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve Rational Equations that Involve Functions

441. For rational function, $f(x)=\frac{x+2}{x^2-6x+8}$, ⓐ find the domain of the function ⓑ solve $f(x)=1$ ⓒ find the points on the graph at this function value.

442. For rational function, $f(x)=\frac{2-x}{x^2+7x+10}$, ⓐ find the domain of the function ⓑ solve $f(x)=2$ ⓒ find the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

443. $\frac{V}{l}=hw$ for l.

444. $\frac{1}{x}-\frac{2}{y}=5$ for y.

445. $x=\frac{y+5}{z-7}$ for z.

446. $P=\frac{k}{V}$ for V.

7.5 Solve Applications with Rational Equations

Solve Proportions

In the following exercises, solve.

447. $\frac{x}{4} = \frac{3}{5}$

448. $\frac{3}{y} = \frac{9}{5}$

449. $\frac{s}{s+20} = \frac{3}{7}$

450. $\frac{t-3}{5} = \frac{t+2}{9}$

Solve Using Proportions

In the following exercises, solve.

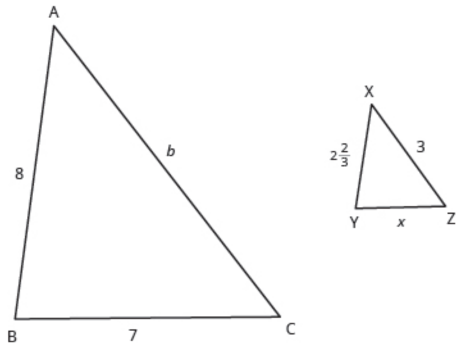
451. Rachael had a 21-ounce strawberry shake that has 739 calories. How many calories are there in a 32-ounce shake?

452. Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

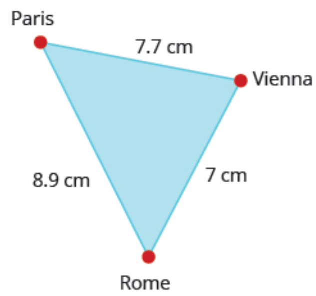
453. ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.



454. On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

ⓐ Paris to Rome

ⓑ Paris to Vienna



455. Francesca is 5.75 feet tall. Late one afternoon, her shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

456. The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5-foot-tall Natasha cast a 1.1-foot shadow. How long would the shadow of the lighthouse be?

Solve Uniform Motion Applications

In the following exercises, solve.

457. When making the 5-hour drive home from visiting her parents, Lolo ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles when it turned bad. How fast did she drive when the weather was bad?

458. Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

459. Josue can ride his bicycle 8 mph faster than Arjun can ride his bike. It takes Luke 3 hours longer than Josue to ride 48 miles. How fast can John ride his bike?

460. Curtis was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

461. Brandy can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

462. Prem takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

463. Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

464. Marta and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Solve Direct Variation Problems

In the following exercises, solve.

465. If y varies directly as x when $y = 9$ and $x = 3,$ find x when $y = 21.$

466. If y varies inversely as x when $y = 20$ and $x = 2,$ find y when $x = 4.$

467. Vanessa is traveling to see her fiancé. The distance, $d,$ varies directly with the speed, $v,$ she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

468. If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $12, how much would a 6” diameter pizza cost?

469. The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Solve Inverse Variation Problems

In the following exercises, solve.

470. If m varies inversely with the square of $n,$ when $m = 4$ and $n = 6$ find m when $n = 2.$

471. The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $6, how many tickets can Madelyn afford to buy if the price increased to $8?

472. On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12-inch string have?

7.6 Solve Rational Inequalities

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

473. $\frac{x-3}{x+4} \le 0$

474. $\frac{5x}{x-2} > 1$

475. $\frac{3x-2}{x-4} \le 2$

476. $\frac{1}{x^2 - 4x - 12} < 0$

477. $\frac{1}{2} - \frac{4}{x^2} \ge \frac{1}{x}$

478. $\frac{4}{x-2} < \frac{3}{x+1}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation

479. Given the function, $R(x) = \frac{x-5}{x-2}$, find the values of x that make the function greater than or equal to 0.

480. Given the function, $R(x) = \frac{x+1}{x+3}$, find the values of x that make the function less than or equal to 0.

481. The function $C(x) = 150x + 100{,}000$ represents the cost to produce x, number of items. Find ⓐ the average cost function, $c(x)$ ⓑ how many items should be produced so that the average cost is less than \$160.

482. Tillman is starting his own business by selling tacos at the beach. Accounting for the cost of his food truck and ingredients for the tacos, the function $C(x) = 2x + 6{,}000$ represents the cost for Tillman to produce x, tacos. Find ⓐ the average cost function, $c(x)$ for Tillman's Tacos ⓑ how many tacos should Tillman produce so that the average cost is less than \$4.

PRACTICE TEST

In the following exercises, simplify.

483. $\dfrac{4a^2b}{12ab^2}$

484. $\dfrac{6x-18}{x^2-9}$

In the following exercises, perform the indicated operation and simplify.

485. $\dfrac{4x}{x+2}\cdot\dfrac{x^2+5x+6}{12x^2}$

486. $\dfrac{2y^2}{y^2-1}\div\dfrac{y^3-y^2+y}{y^3-1}$

487. $\dfrac{6x^2-x+20}{x^2-81}-\dfrac{5x^2+11x-7}{x^2-81}$

488. $\dfrac{-3a}{3a-3}+\dfrac{5a}{a^2+3a-4}$

489. $\dfrac{2n^2+8n-1}{n^2-1}-\dfrac{n^2-7n-1}{1-n^2}$

490. $\dfrac{10x^2+16x-7}{8x-3}+\dfrac{2x^2+3x-1}{3-8x}$

491. $\dfrac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

In the following exercises, solve each equation.

492. $\frac{1}{x}+\frac{3}{4}=\frac{5}{8}$

493. $\dfrac{1}{z-5}+\dfrac{1}{z+5}=\dfrac{1}{z^2-25}$

494. $\dfrac{z}{2z+8}-\dfrac{3}{4z-8}=\dfrac{3z^2-16z-16}{8z^2+2z-64}$

In the following exercises, solve each rational inequality and write the solution in interval notation.

495. $\dfrac{6x}{x-6}\le 2$

496. $\dfrac{2x+3}{x-6}>1$

497. $\frac{1}{2}+\frac{12}{x^2}\ge\frac{5}{x}$

In the following exercises, find $R(x)$ given $f(x)=\dfrac{x-4}{x^2-3x-10}$ and $g(x)=\dfrac{x-5}{x^2-2x-8}$.

498. $R(x)=f(x)-g(x)$

499. $R(x)=f(x)\cdot g(x)$

500. $R(x)=f(x)\div g(x)$

501. Given the function, $R(x)=\dfrac{2}{2x^2+x-15}$, find the values of x that make the function less than or equal to 0.

In the following exercises, solve.

502. If y varies directly with x, and $x=5$ when $y=30$, find x when $y=42$.

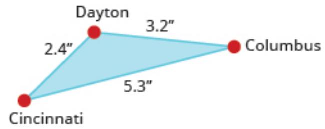
503. If y varies inversely with the square of x and $x=3$ when $y=9$, find y when $x=4$.

504. Matheus can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Matheus' speed on his bike?

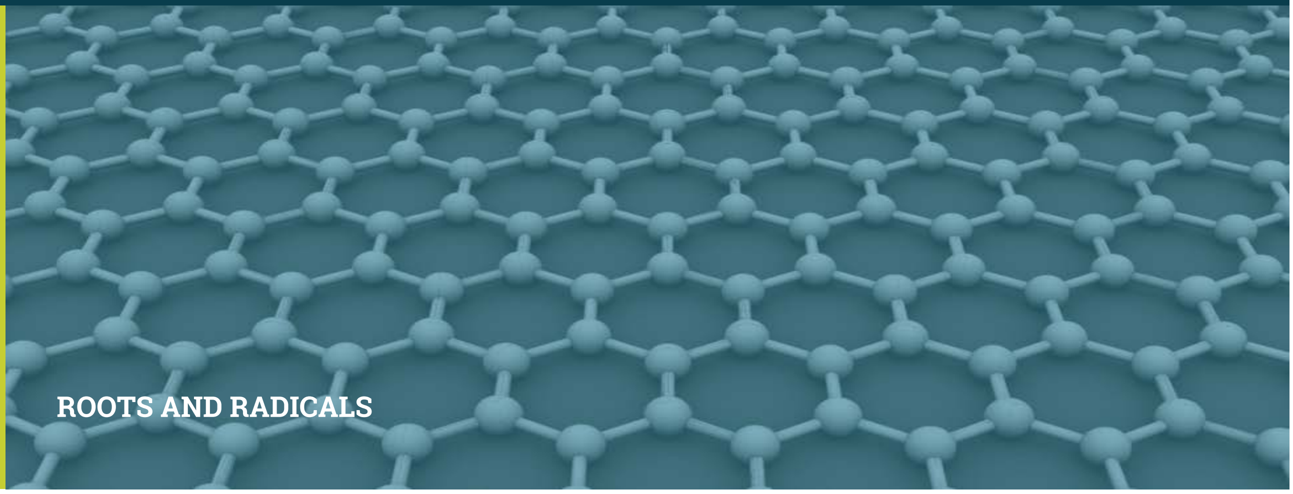
505. Oliver can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Oliver’s dad working alone to split the logs?

506. The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

507. The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio. The diagram gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?



8 ROOTS AND RADICALS

Figure 8.1 Graphene is an incredibly strong and flexible material made from carbon. It can also conduct electricity. Notice the hexagonal grid pattern. (credit: “AlexanderAIUS” / Wikimedia Commons)

Chapter Outline

Introduction

Imagine charging your cell phone is less than five seconds. Consider cleaning radioactive waste from contaminated water. Think about filtering salt from ocean water to make an endless supply of drinking water. Ponder the idea of bionic devices that can repair spinal injuries. These are just of few of the many possible uses of a material called graphene. Materials scientists are developing a material made up of a single layer of carbon atoms that is stronger than any other material, completely flexible, and conducts electricity better than most metals. Research into this type of material requires a solid background in mathematics, including understanding roots and radicals. In this chapter, you will learn to simplify expressions containing roots and radicals, perform operations on radical expressions and equations, and evaluate radical functions.

8.1 Simplify Expressions with Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: ⓐ $(-9)^2$ ⓑ -9^2 ⓒ $(-9)^3$.
   If you missed this problem, review **Example 2.21**.

2. Round 3.846 to the nearest hundredth.
   If you missed this problem, review **Example 1.34**.
3. Simplify: ⓐ $x^3 \cdot x^3$ ⓑ $y^2 \cdot y^2 \cdot y^2$ ⓒ $z^3 \cdot z^3 \cdot z^3 \cdot z^3$.
   If you missed this problem, review **Example 5.12**.

Simplify Expressions with Roots

In **Foundations**, we briefly looked at square roots. Remember that when a real number *n* is multiplied by itself, we write n^2 and read it '*n* squared'. This number is called the **square** of *n*, and *n* is called the **square root**. For example,

$$13^2 \text{ is read "13 squared"}$$
$$169 \text{ is called the } \textit{square} \text{ of 13, since } 13^2 = 169$$
$$13 \text{ is a } \textit{square root} \text{ of } 169$$

Square and Square Root of a number

Square

$$\text{If } n^2 = m, \text{ then } m \text{ is the } \textbf{square} \text{ of } n.$$

Square Root

$$\text{If } n^2 = m, \text{ then } n \text{ is a } \textbf{square root} \text{ of } m.$$

Notice $(-13)^2 = 169$ also, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of *m*. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Square Root Notation

$$\sqrt{m} \text{ is read "the square root of } m\text{".}$$
$$\text{If } n^2 = m, \text{ then } n = \sqrt{m}, \text{ for } n \geq 0.$$

radical sign → $\sqrt{m}$ ← radicand

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

EXAMPLE 8.1

Simplify: ⓐ $\sqrt{144}$ ⓑ $-\sqrt{289}$.

Solution

ⓐ

	$\sqrt{144}$
Since $12^2 = 144$.	12

ⓑ

	$-\sqrt{289}$
Since $17^2 = 289$ and the negative is in front of the radical sign.	-17

TRY IT : : 8.1 Simplify: ⓐ $-\sqrt{64}$ ⓑ $\sqrt{225}$.

TRY IT : : 8.2 Simplify: ⓐ $\sqrt{100}$ ⓑ $-\sqrt{121}$.

Can we simplify $\sqrt{-49}$? Is there a number whose square is -49?

$$(\ )^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

EXAMPLE 8.2

Simplify: ⓐ $\sqrt{-196}$ ⓑ $-\sqrt{64}$.

Solution

ⓐ

	$\sqrt{-196}$
There is no real number whose square is -196.	$\sqrt{-196}$ is not a real number.

ⓑ

	$-\sqrt{64}$
The negative is in front of the radical.	-8

TRY IT : : 8.3 Simplify: ⓐ $\sqrt{-169}$ ⓑ $-\sqrt{81}$.

TRY IT : : 8.4 Simplify: ⓐ $-\sqrt{49}$ ⓑ $\sqrt{-121}$.

So far we have only talked about squares and square roots. Let's now extend our work to include higher powers and higher roots.

Let's review some vocabulary first.

We write:	We say:
n^2	n squared
n^3	n cubed
n^4	n to the fourth power
n^5	n to the fi th power

The terms 'squared' and 'cubed' come from the formulas for area of a square and volume of a cube.

It will be helpful to have a table of the powers of the integers from −5 to 5. See **Figure 8.2**.

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	1024
5	25	125	625	3125
x	x^2	x^3	x^4	x^5
x^2	x^4	x^6	x^8	x^{10}

Number	Square	Cube	Fourth power	Fifth power
n	n^2	n^3	n^4	n^5
−1	1	−1	1	−1
−2	4	−8	16	−32
−3	9	−27	81	−243
−4	16	−64	256	−1024
−5	25	−125	625	−3125

Figure 8.2

Notice the signs in the table. All powers of positive numbers are positive, of course. But when we have a negative number, the *even* powers are positive and the *odd* powers are negative. We'll copy the row with the powers of −2 to help you see this.

n	n^2	n^3	n^4	n^5
–2	4	–8	16	–32

Even power
Positive result

Odd power
Negative result

We will now extend the square root definition to higher roots.

n^{th} Root of a Number

$$\text{If } b^n = a, \text{ then } b \text{ is an } n^{th} \text{ root of } a.$$
$$\text{The principal } n^{th} \text{ root of } a \text{ is written } \sqrt[n]{a}.$$
$$n \text{ is called the } \textbf{index} \text{ of the radical.}$$

Just like we use the word 'cubed' for b^3, we use the term 'cube root' for $\sqrt[3]{a}$.

We can refer to **Figure 8.2** to help find higher roots.

$$\begin{aligned} 4^3 &= 64 \\ 3^4 &= 81 \\ (-2)^5 &= -32 \end{aligned} \qquad \begin{aligned} \sqrt[3]{64} &= 4 \\ \sqrt[4]{81} &= 3 \\ \sqrt[5]{-32} &= -2 \end{aligned}$$

Could we have an even root of a negative number? We know that the square root of a negative number is not a real number. The same is true for any even root. *Even* roots of negative numbers are not real numbers. *Odd* roots of negative numbers are real numbers.

Properties of $\sqrt[n]{a}$

When *n* is an even number and

- $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
- $a < 0$, then $\sqrt[n]{a}$ is not a real number.

When *n* is an odd number, $\sqrt[n]{a}$ is a real number for all values of *a*.

We will apply these properties in the next two examples.

EXAMPLE 8.3

Simplify: ⓐ $\sqrt[3]{64}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{32}$.

Solution

ⓐ

	$\sqrt[3]{64}$
Since $4^3 = 64$.	4

ⓑ

	$\sqrt[4]{81}$
Since $(3)^4 = 81$.	3

ⓒ

	$\sqrt[5]{32}$
Since $(2)^5 = 32$.	2

TRY IT : : 8.5 Simplify: ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{256}$ ⓒ $\sqrt[5]{243}$.

TRY IT : : 8.6 Simplify: ⓐ $\sqrt[3]{1000}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$.

In this example be alert for the negative signs as well as even and odd powers.

EXAMPLE 8.4

Simplify: ⓐ $\sqrt[3]{-125}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{-243}$.

Solution

ⓐ

	$\sqrt[3]{-125}$
Since $(-5)^3 = -125$.	-5

ⓑ

	$\sqrt[4]{-16}$
Think, $(\,?\,)^4 = -16$. No real number raised to the fourth power is negative.	Not a real number.

ⓒ

	$\sqrt[5]{-243}$
Since $(-3)^5 = -243$.	-3

TRY IT : : 8.7 Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-256}$ ⓒ $\sqrt[5]{-32}$.

TRY IT : : 8.8 Simplify: ⓐ $\sqrt[3]{-216}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1024}$.

Estimate and Approximate Roots

When we see a number with a radical sign, we often don't think about its numerical value. While we probably know that the $\sqrt{4} = 2$, what is the value of $\sqrt{21}$ or $\sqrt[3]{50}$? In some situations a quick estimate is meaningful and in others it is convenient to have a decimal approximation.

To get a numerical estimate of a square root, we look for perfect square numbers closest to the radicand. To find an estimate of $\sqrt{11}$, we see 11 is between perfect square numbers 9 and 16, *closer* to 9. Its square root then will be between 3 and 4, but closer to 3.

Number	Square Root
4	2
9	3
16	4
25	5

$9 < 11 < 16$

$3 < \sqrt{11} < 4$

Number	Cube Root
8	2
27	3
64	4
125	5

$64 < 91 < 125$

$4 < \sqrt[3]{91} < 5$

Similarly, to estimate $\sqrt[3]{91}$, we see 91 is between perfect cube numbers 64 and 125. The cube root then will be between 4 and 5.

EXAMPLE 8.5

Estimate each root between two consecutive whole numbers: ⓐ $\sqrt{105}$ ⓑ $\sqrt[3]{43}$.

Solution

ⓐ Think of the perfect square numbers closest to 105. Make a small table of these perfect squares and their squares roots.

$\sqrt{105}$

Number	Square Root
81	9
100	10
121	11
144	12

105 $\sqrt{105}$

Locate 105 between two consecutive perfect squares.	$100 < 105 < 121$
$\sqrt{105}$ is between their square roots.	$10 < \sqrt{105} < 11$

ⓑ Similarly we locate 43 between two perfect cube numbers.

$\sqrt[3]{43}$

Number	Cube Root
8	2
27	3
64	4
125	5

43 $\sqrt[3]{43}$

Locate 43 between two consecutive perfect cubes.	$27 < 43 < 64$
$\sqrt[3]{43}$ is between their cube roots.	$3 < \sqrt[3]{43} < 4$

TRY IT : : 8.9 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{38}$ ⓑ $\sqrt[3]{93}$

TRY IT : : 8.10 Estimate each root between two consecutive whole numbers:

ⓐ $\sqrt{84}$ ⓑ $\sqrt[3]{152}$

There are mathematical methods to approximate square roots, but nowadays most people use a calculator to find square roots. To find a square root you will use the $\sqrt{x}$ key on your calculator. To find a cube root, or any root with higher index, you will use the $\sqrt[y]{x}$ key.

When you use these keys, you get an approximate value. It is an approximation, accurate to the number of digits shown on your calculator's display. The symbol for an approximation is $\approx$ and it is read 'approximately'.

Suppose your calculator has a 10 digit display. You would see that

$$\sqrt{5} \approx 2.236067978 \text{ rounded to two decimal places is } \sqrt{5} \approx 2.24$$

$$\sqrt[4]{93} \approx 3.105422799 \text{ rounded to two decimal places is } \sqrt[4]{93} \approx 3.11$$

How do we know these values are approximations and not the exact values? Look at what happens when we square them:

$$\begin{aligned} (2.236067978)^2 &= 5.000000002 \qquad & (3.105422799)^4 &= 92.999999991 \\ (2.24)^2 &= 5.0176 & (3.11)^4 &= 93.54951841 \end{aligned}$$

Their squares are close to 5, but are not exactly equal to 5. The fourth powers are close to 93, but not equal to 93.

EXAMPLE 8.6

Round to two decimal places: ⓐ $\sqrt{17}$ ⓑ $\sqrt[3]{49}$ ⓒ $\sqrt[4]{51}$.

Solution

ⓐ

	$\sqrt{17}$
Use the calculator square root key.	$4.123105626\ldots$
Round to two decimal places.	4.12
	$\sqrt{17} \approx 4.12$

ⓑ

	$\sqrt[3]{49}$
Use the calculator $\sqrt[y]{x}$ key.	$3.659305710\ldots$
Round to two decimal places.	3.66
	$\sqrt[3]{49} \approx 3.66$

ⓒ

	$\sqrt[4]{51}$
Use the calculator $\sqrt[y]{x}$ key.	$2.6723451177\ldots$
Round to two decimal places.	2.67
	$\sqrt[4]{51} \approx 2.67$

TRY IT : : 8.11 Round to two decimal places:

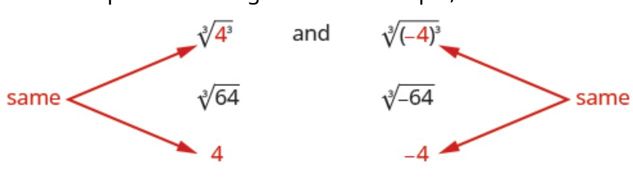
ⓐ $\sqrt{11}$ ⓑ $\sqrt[3]{71}$ ⓒ $\sqrt[4]{127}$.

TRY IT : : 8.12 Round to two decimal places:

ⓐ $\sqrt{13}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{98}$.

Simplify Variable Expressions with Roots

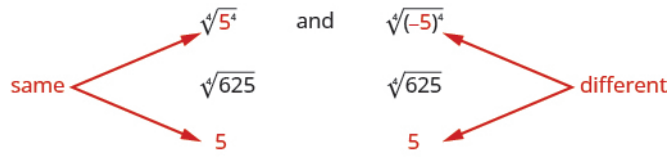
The odd root of a number can be either positive or negative. For example,



In either case, when n is odd, $\sqrt[n]{a^n} = a$.

But what about an even root? We want the principal root, so $\sqrt[4]{625} = 5$.

But notice,



Here we see that sometimes when n is even, $\sqrt[n]{a^n} \neq a$.

How can we make sure the fourth root of −5 raised to the fourth power is 5? We can use the absolute value. $|-5| = 5$. So we say that when n is even $\sqrt[n]{a^n} = |a|$. This guarantees the principal root is positive.

Simplifying Odd and Even Roots

For any integer $n \geq 2$,

when the index n is odd	$\sqrt[n]{a^n} = a$
when the index n is even	$\sqrt[n]{a^n} = \lvert a \rvert$

We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

EXAMPLE 8.7

Simplify: ⓐ $\sqrt{x^2}$ ⓑ $\sqrt[3]{n^3}$ ⓒ $\sqrt[4]{p^4}$ ⓓ $\sqrt[5]{y^5}$.

Solution

ⓐ We use the absolute value to be sure to get the positive root.

	$\sqrt{x^2}$
Since the index n is even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\lvert x \rvert$

ⓑ This is an odd indexed root so there is no need for an absolute value sign.

	$\sqrt[3]{m^3}$
Since the index n is odd, $\sqrt[n]{a^n} = a$.	m

ⓒ

	$\sqrt[4]{p^4}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$\lvert p \rvert$

ⓓ

	$\sqrt[5]{y^5}$
Since the index n is odd, $\sqrt[n]{a^n} = a$.	y

TRY IT : : 8.13 Simplify: ⓐ $\sqrt{b^2}$ ⓑ $\sqrt[3]{w^3}$ ⓒ $\sqrt[4]{m^4}$ ⓓ $\sqrt[5]{q^5}$.

TRY IT : : 8.14 Simplify: ⓐ $\sqrt{y^2}$ ⓑ $\sqrt[3]{p^3}$ ⓒ $\sqrt[4]{z^4}$ ⓓ $\sqrt[5]{q^5}$.

What about square roots of higher powers of variables? The Power Property of Exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root,

	$\sqrt{a^{2m}}$
Since $(a^m)^2 = a^{2m}$.	$\sqrt{(a^m)^2}$
Since n is even $\sqrt[n]{a^n} = \|a\|$.	$\|a^m\|$

$$\text{So } \sqrt{a^{2m}} = |a^m|.$$

We apply this concept in the next example.

EXAMPLE 8.8

Simplify: ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$.

Solution

ⓐ

	$\sqrt{x^6}$
Since $\left(x^3\right)^2 = x^6$.	$\sqrt{\left(x^3\right)^2}$
Since the index n is even $\sqrt{a^n} = \|a\|$.	$\left\|x^3\right\|$

ⓑ

	$\sqrt{y^{16}}$
Since $\left(y^8\right)^2 = y^{16}$.	$\sqrt{\left(y^8\right)^2}$
Since the index n is even $\sqrt[n]{a^n} = \|a\|$.	y^8
In this case the absolute value sign is not needed as y^8 is positive.	

TRY IT : : 8.15 Simplify: ⓐ $\sqrt{y^{18}}$ ⓑ $\sqrt{z^{12}}$.

TRY IT : : 8.16 Simplify: ⓐ $\sqrt{m^4}$ ⓑ $\sqrt{b^{10}}$.

The next example uses the same idea for highter roots.

EXAMPLE 8.9

Simplify: ⓐ $\sqrt[3]{y^{18}}$ ⓑ $\sqrt[4]{z^8}$.

Solution

ⓐ

	$\sqrt[3]{y^{18}}$
Since $\left(y^6\right)^3 = y^{18}$.	$\sqrt[3]{\left(y^6\right)^3}$
Since n is odd, $\sqrt[n]{a^n} = a$.	y^6

ⓑ

	$\sqrt[4]{z^8}$
Since $\left(z^2\right)^4 = z^8$.	$\sqrt[4]{\left(z^2\right)^4}$
Since z^2 is positive, we do not need an absolute value sign.	z^2

> **TRY IT : : 8.17** Simplify: ⓐ $\sqrt[4]{u^{12}}$ ⓑ $\sqrt[3]{v^{15}}$.

> **TRY IT : : 8.18** Simplify: ⓐ $\sqrt[5]{c^{20}}$ ⓑ $\sqrt[6]{d^{24}}$

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = |a^m|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always positive.

EXAMPLE 8.10

Simplify: ⓐ $\sqrt{16n^2}$ ⓑ $-\sqrt{81c^2}$.

Solution

ⓐ

	$\sqrt{16n^2}$
Since $(4n)^2 = 16n^2$.	$\sqrt{(4n)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$4\lvert n \rvert$

ⓑ

	$-\sqrt{81c^2}$
Since $(9c)^2 = 81c^2$.	$-\sqrt{(9c)^2}$
Since the index n is even $\sqrt[n]{a^n} = \lvert a \rvert$.	$-9\lvert c \rvert$

> **TRY IT : : 8.19** Simplify: ⓐ $\sqrt{64x^2}$ ⓑ $-\sqrt{100p^2}$.

> **TRY IT : : 8.20** Simplify: ⓐ $\sqrt{169y^2}$ ⓑ $-\sqrt{121y^2}$.

This example just takes the idea farther as it has roots of higher index.

EXAMPLE 8.11

Simplify: ⓐ $\sqrt[3]{64p^6}$ ⓑ $\sqrt[4]{16q^{12}}$.

Solution

ⓐ

	$\sqrt[3]{64p^6}$
Rewrite $64p^6$ as $\left(4p^2\right)^3$.	$\sqrt[3]{\left(4p^2\right)^3}$
Take the cube root.	$4p^2$

ⓑ

	$\sqrt[4]{16q^{12}}$
Rewrite the radicand as a fourth power.	$\sqrt[4]{\left(2q^3\right)^4}$
Take the fourth root.	$2\left\lvert q^3\right\rvert$

TRY IT : : 8.21 Simplify: ⓐ $\sqrt[3]{27x^{27}}$ ⓑ $\sqrt[4]{81q^{28}}$.

TRY IT : : 8.22 Simplify: ⓐ $\sqrt[3]{125q^9}$ ⓑ $\sqrt[5]{243q^{25}}$.

The next examples have two variables.

EXAMPLE 8.12

Simplify: ⓐ $\sqrt{36x^2y^2}$ ⓑ $\sqrt{121a^6b^8}$ ⓒ $\sqrt[3]{64p^{63}q^9}$.

Solution

ⓐ

	$\sqrt{36x^2y^2}$
Since $(6xy)^2 = 36x^2y^2$	$\sqrt{(6xy)^2}$
Take the square root.	$6\lvert xy\rvert$

ⓑ

	$\sqrt{121a^6b^8}$
Since $\left(11a^3b^4\right)^2 = 121a^6b^8$	$\sqrt{\left(11a^3b^4\right)^2}$
Take the square root.	$11\left\lvert a^3\right\rvert b^4$

ⓒ

	$\sqrt[3]{64p^{63}q^9}$
Since $\left(4p^{21}q^3\right)^3 = 64p^{63}q^9$	$\sqrt[3]{\left(4p^{21}q^3\right)^3}$
Take the cube root.	$4p^{21}q^3$

TRY IT : : 8.23 Simplify: ⓐ $\sqrt{100a^2 b^2}$ ⓑ $\sqrt{144p^{12} q^{20}}$ ⓒ $\sqrt[3]{8x^{30} y^{12}}$

TRY IT : : 8.24 Simplify: ⓐ $\sqrt{225m^2 n^2}$ ⓑ $\sqrt{169x^{10} y^{14}}$ ⓒ $\sqrt[3]{27w^{36} z^{15}}$

MEDIA : :

Access this online resource for additional instruction and practice with simplifying expressions with roots.

- **Simplifying Variables Exponents with Roots using Absolute Values (https://openstax.org/l/37SimVarAbVal)**

8.1 EXERCISES

Practice Makes Perfect

Simplify Expressions with Roots

In the following exercises, simplify.

1. ⓐ $\sqrt{64}$ ⓑ $-\sqrt{81}$

2. ⓐ $\sqrt{169}$ ⓑ $-\sqrt{100}$

3. ⓐ $\sqrt{196}$ ⓑ $-\sqrt{1}$

4. ⓐ $\sqrt{144}$ ⓑ $-\sqrt{121}$

5. ⓐ $\sqrt{\frac{4}{9}}$ ⓑ $-\sqrt{0.01}$

6. ⓐ $\sqrt{\frac{64}{121}}$ ⓑ $-\sqrt{0.16}$

7. ⓐ $\sqrt{-121}$ ⓑ $-\sqrt{289}$

8. ⓐ $-\sqrt{400}$ ⓑ $\sqrt{-36}$

9. ⓐ $-\sqrt{225}$ ⓑ $\sqrt{-9}$

10. ⓐ $\sqrt{-49}$ ⓑ $-\sqrt{256}$

11. ⓐ $\sqrt[3]{216}$ ⓑ $\sqrt[4]{256}$

12. ⓐ $\sqrt[3]{27}$ ⓑ $\sqrt[4]{16}$ ⓒ $\sqrt[5]{243}$

13. ⓐ $\sqrt[3]{512}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{1}$

14. ⓐ $\sqrt[3]{125}$ ⓑ $\sqrt[4]{1296}$ ⓒ $\sqrt[5]{1024}$

15. ⓐ $\sqrt[3]{-8}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-32}$

16.
ⓐ $\sqrt[3]{-64}$
ⓑ $\sqrt[4]{-16}$
ⓒ $\sqrt[5]{-243}$

17.
ⓐ $\sqrt[3]{-125}$
ⓑ $\sqrt[4]{-1296}$
ⓒ $\sqrt[5]{-1024}$

18.
ⓐ $\sqrt[3]{-512}$
ⓑ $\sqrt[4]{-81}$
ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

19. ⓐ $\sqrt{70}$ ⓑ $\sqrt[3]{71}$

20. ⓐ $\sqrt{55}$ ⓑ $\sqrt[3]{119}$

21. ⓐ $\sqrt{200}$ ⓑ $\sqrt[3]{137}$

22. ⓐ $\sqrt{172}$ ⓑ $\sqrt[3]{200}$

In the following exercises, approximate each root and round to two decimal places.

23. ⓐ $\sqrt{19}$ ⓑ $\sqrt[3]{89}$ ⓒ $\sqrt[4]{97}$

24. ⓐ $\sqrt{21}$ ⓑ $\sqrt[3]{93}$ ⓒ $\sqrt[4]{101}$

25. ⓐ $\sqrt{53}$ ⓑ $\sqrt[3]{147}$ ⓒ $\sqrt[4]{452}$

26. ⓐ $\sqrt{47}$ ⓑ $\sqrt[3]{163}$ ⓒ $\sqrt[4]{527}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

27. ⓐ $\sqrt[5]{u^5}$ ⓑ $\sqrt[8]{v^8}$

28. ⓐ $\sqrt[3]{a^3}$ ⓑ $\sqrt[9]{b^9}$

29. ⓐ $\sqrt[4]{y^4}$ ⓑ $\sqrt[7]{m^7}$

30. ⓐ $\sqrt[8]{k^8}$ ⓑ $\sqrt[6]{p^6}$

31. ⓐ $\sqrt{x^6}$ ⓑ $\sqrt{y^{16}}$

32. ⓐ $\sqrt{a^{14}}$ ⓑ $\sqrt{w^{24}}$

33. ⓐ $\sqrt{x^{24}}$ ⓑ $\sqrt{y^{22}}$

34. ⓐ $\sqrt{a^{12}}$ ⓑ $\sqrt{b^{26}}$

35. ⓐ $\sqrt[3]{x^9}$ ⓑ $\sqrt[4]{y^{12}}$

36. ⓐ $\sqrt[5]{a^{10}}$ ⓑ $\sqrt[3]{b^{27}}$

37. ⓐ $\sqrt[4]{m^8}$ ⓑ $\sqrt[5]{n^{20}}$

38. ⓐ $\sqrt[6]{r^{12}}$ ⓑ $\sqrt[3]{s^{30}}$

39. ⓐ $\sqrt{49x^2}$ ⓑ $-\sqrt{81x^{18}}$

40. ⓐ $\sqrt{100y^2}$ ⓑ $-\sqrt{100m^{32}}$

41. ⓐ $\sqrt{121m^{20}}$ ⓑ $-\sqrt{64a^2}$

42.
ⓐ $\sqrt{81x^{36}}$
ⓑ $-\sqrt{25x^2}$

43.
ⓐ $\sqrt[4]{16x^8}$
ⓑ $\sqrt[6]{64y^{12}}$

44.
ⓐ $\sqrt[3]{-8c^9}$
ⓑ $\sqrt[3]{125d^{15}}$

45.
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

46.
ⓐ $\sqrt[7]{128r^{14}}$
ⓑ $\sqrt[4]{81s^{24}}$

47.
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

48.
ⓐ $\sqrt{196a^2b^2}$
ⓑ $\sqrt{81p^{24}q^6}$
ⓒ $\sqrt[3]{27p^{45}q^9}$

49.
ⓐ $\sqrt{121a^2b^2}$
ⓑ $\sqrt{9c^8d^{12}}$
ⓒ $\sqrt[3]{64x^{15}y^{66}}$

50.
ⓐ $\sqrt{225x^2y^2z^2}$
ⓑ $\sqrt{36r^6s^{20}}$
ⓒ $\sqrt[3]{125y^{18}z^{27}}$

Writing Exercises

51. Why is there no real number equal to $\sqrt{-64}$?

52. What is the difference between 9^2 and $\sqrt{9}$?

53. Explain what is meant by the n^{th} root of a number.

54. Explain the difference of finding the n^{th} root of a number when the index is even compared to when the index is odd.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

8.2 Simplify Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{x^9}{x^4}$.
   If you missed this problem, review Example 5.13.
2. Simplify: $\frac{y^3}{y^{11}}$.
   If you missed this problem, review Example 5.13.
3. Simplify: $\left(n^2\right)^6$.
   If you missed this problem, review Example 5.17.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Simplified Radical Expression

For real numbers *a* and *m*, and $n \geq 2$,

$$\sqrt[n]{a} \text{ is considered simplified i } \; a \text{ has no factors of } m^n$$

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

Similarly, $\sqrt[3]{4}$ is simplified because there are no perfect cube factors in 4. But $\sqrt[3]{24}$ is not simplified because 24 has a perfect cube factor of 8.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that $(ab)^n = a^n b^n$. The corresponding of **Product Property of Roots** says that $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

Product Property of n^{th} Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

We use the Product Property of Roots to remove all perfect square factors from a square root.

EXAMPLE 8.13 SIMPLIFY SQUARE ROOTS USING THE PRODUCT PROPERTY OF ROOTS

Simplify: $\sqrt{98}$.

Solution

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.	We see that 49 is the largest factor of 98 that has a power of 2. In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{98}$ $\sqrt{49 \cdot 2}$
Step 2. Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Step 3. Simplify the root of the perfect power.		$7\sqrt{2}$

TRY IT : : 8.25 Simplify: $\sqrt{48}$.

TRY IT : : 8.26 Simplify: $\sqrt{45}$.

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index. The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

HOW TO : : SIMPLIFY A RADICAL EXPRESSION USING THE PRODUCT PROPERTY.

Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.

Step 2. Use the product rule to rewrite the radical as the product of two radicals.

Step 3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares, cubes, and fourth powers.

EXAMPLE 8.14

Simplify: ⓐ $\sqrt{500}$ ⓑ $\sqrt[3]{16}$ ⓒ $\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{500}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{100 \cdot 5}$
Rewrite the radical as the product of two radicals	$\sqrt{100} \cdot \sqrt{5}$
Simplify.	$10\sqrt{5}$

ⓑ

	$\sqrt[3]{16}$
Rewrite the radicand as a product using the greatest perfect cube factor. $2^3 = 8$	$\sqrt[3]{8 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8} \cdot \sqrt[3]{2}$
Simplify.	$2\sqrt[3]{2}$

ⓒ

	$\sqrt[4]{243}$
Rewrite the radicand as a product using the greatest perfect fourth power factor. $3^4 = 81$	$\sqrt[4]{81 \cdot 3}$
Rewrite the radical as the product of two radicals	$\sqrt[4]{81} \cdot \sqrt[4]{3}$
Simplify.	$3\sqrt[4]{3}$

TRY IT : : 8.27 Simplify: ⓐ $\sqrt{288}$ ⓑ $\sqrt[3]{81}$ ⓒ $\sqrt[4]{64}$.

TRY IT : : 8.28 Simplify: ⓐ $\sqrt{432}$ ⓑ $\sqrt[3]{625}$ ⓒ $\sqrt[4]{729}$.

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

EXAMPLE 8.15

Simplify: ⓐ $\sqrt{x^3}$ ⓑ $\sqrt[3]{x^4}$ ⓒ $\sqrt[4]{x^7}$.

Solution

ⓐ

	$\sqrt{x^3}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{x^2 \cdot x}$
Rewrite the radical as the product of two radicals.	$\sqrt{x^2} \cdot \sqrt{x}$
Simplify.	$\lvert x \rvert \sqrt{x}$

ⓑ

	$\sqrt[3]{x^4}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{x^3 \cdot x}.$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{x^3} \cdot \sqrt[3]{x}$
Simplify.	$x\sqrt[3]{x}$

ⓒ

	$\sqrt[4]{x^7}$
Rewrite the radicand as a product using the greatest perfect fourth power factor.	$\sqrt[4]{x^4 \cdot x^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{x^4} \cdot \sqrt[4]{x^3}$
Simplify.	$\lvert x \rvert \sqrt[4]{x^3}$

TRY IT : : 8.29 Simplify: ⓐ $\sqrt{b^5}$ ⓑ $\sqrt[4]{y^6}$ ⓒ $\sqrt[3]{z^5}$

TRY IT : : 8.30 Simplify: ⓐ $\sqrt{p^9}$ ⓑ $\sqrt[5]{y^8}$ ⓒ $\sqrt[6]{q^{13}}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

EXAMPLE 8.16

Simplify: ⓐ $\sqrt{72n^7}$ ⓑ $\sqrt[3]{24x^7}$ ⓒ $\sqrt[4]{80y^{14}}$.

Solution

ⓐ

	$\sqrt{72n^7}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{36n^6 \cdot 2n}$
Rewrite the radical as the product of two radicals.	$\sqrt{36n^6} \cdot \sqrt{2n}$
Simplify.	$6\lvert n^3 \rvert \sqrt{2n}$

ⓑ

	$\sqrt[3]{24x^7}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{8x^6 \cdot 3x}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8x^6} \cdot \sqrt[3]{3x}$
Rewrite the fir t radicand as $\left(2x^2\right)^3$.	$\sqrt[3]{\left(2x^2\right)^3} \cdot \sqrt[3]{3x}$
Simplify.	$2x^2 \sqrt[3]{3x}$

ⓒ

	$\sqrt[4]{80y^{14}}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{16y^{12} \cdot 5y^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{16y^{12}} \cdot \sqrt[4]{5y^2}$
Rewrite the fir t radicand as $\left(2y^3\right)^4$.	$\sqrt[4]{\left(2y^3\right)^4} \cdot \sqrt[4]{5y^2}$
Simplify.	$2\left\|y^3\right\|\sqrt[4]{5y^2}$

TRY IT : : 8.31 Simplify: ⓐ $\sqrt{32y^5}$ ⓑ $\sqrt[3]{54p^{10}}$ ⓒ $\sqrt[4]{64q^{10}}$.

TRY IT : : 8.32 Simplify: ⓐ $\sqrt{75a^9}$ ⓑ $\sqrt[3]{128m^{11}}$ ⓒ $\sqrt[4]{162n^7}$.

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

EXAMPLE 8.17

Simplify: ⓐ $\sqrt{63u^3v^5}$ ⓑ $\sqrt[3]{40x^4y^5}$ ⓒ $\sqrt[4]{48x^4y^7}$.

Solution

ⓐ

	$\sqrt{63u^3v^5}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9u^2v^4 \cdot 7uv}$
Rewrite the radical as the product of two radicals.	$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$
Rewrite the fir t radicand as $\left(3uv^2\right)^2$.	$\sqrt{\left(3uv^2\right)^2} \cdot \sqrt{7uv}$
Simplify.	$3\|u\|v^2\sqrt{7uv}$

ⓑ

	$\sqrt[3]{40x^4y^5}$
Rewrite the radicand as a product using the largest perfect cube factor.	$\sqrt[3]{8x^3y^3 \cdot 5xy^2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{8x^3y^3} \cdot \sqrt[3]{5xy^2}$
Rewrite the fir t radicand as $(2xy)^3$.	$\sqrt[3]{(2xy)^3} \cdot \sqrt[3]{5xy^2}$
Simplify.	$2xy\sqrt[3]{5xy^2}$

ⓒ

	$\sqrt[4]{48x^4y^7}$
Rewrite the radicand as a product using the largest perfect fourth power factor.	$\sqrt[4]{16x^4y^4 \cdot 3y^3}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{16x^4y^4} \cdot \sqrt[4]{3y^3}$
Rewrite the fir t radicand as $(2xy)^4$.	$\sqrt[4]{(2xy)^4} \cdot \sqrt[4]{3y^3}$
Simplify.	$2\lvert xy\rvert\sqrt[4]{3y^3}$

TRY IT : : 8.33 Simplify: ⓐ $\sqrt{98a^7b^5}$ ⓑ $\sqrt[3]{56x^5y^4}$ ⓒ $\sqrt[4]{32x^5y^8}$.

TRY IT : : 8.34 Simplify: ⓐ $\sqrt{180m^9n^{11}}$ ⓑ $\sqrt[3]{72x^6y^5}$ ⓒ $\sqrt[4]{80x^7y^4}$.

EXAMPLE 8.18

Simplify: ⓐ $\sqrt[3]{-27}$ ⓑ $\sqrt[4]{-16}$.

Solution

ⓐ

	$\sqrt[3]{-27}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3}$
Take the cube root.	-3

ⓑ

	$\sqrt[4]{-16}$
There is no real number n where $n^4 = -16$.	Not a real number.

TRY IT : : 8.35 Simplify: ⓐ $\sqrt[3]{-64}$ ⓑ $\sqrt[4]{-81}$.

TRY IT : : 8.36 Simplify: ⓐ $\sqrt[3]{-625}$ ⓑ $\sqrt[4]{-324}$.

We have seen how to use the order of operations to simplify some expressions with radicals. In the next example, we have the sum of an integer and a square root. We simplify the square root but cannot add the resulting expression to the integer since one term contains a radical and the other does not. The next example also includes a fraction with a radical in the numerator. Remember that in order to simplify a fraction you need a common factor in the numerator and denominator.

EXAMPLE 8.19

Simplify: ⓐ $3 + \sqrt{32}$ ⓑ $\dfrac{4 - \sqrt{48}}{2}$.

Solution

ⓐ

	$3+\sqrt{32}$
Rewrite the radicand as a product using the largest perfect square factor.	$3+\sqrt{16\cdot 2}$
Rewrite the radical as the product of two radicals.	$3+\sqrt{16}\cdot\sqrt{2}$
Simplify.	$3+4\sqrt{2}$

The terms cannot be added as one has a radical and the other does not. Trying to add an integer and a radical is like trying to add an integer and a variable. They are not like terms!

ⓑ

	$\dfrac{4-\sqrt{48}}{2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\dfrac{4-\sqrt{16\cdot 3}}{2}$
Rewrite the radical as the product of two radicals.	$\dfrac{4-\sqrt{16}\cdot\sqrt{3}}{2}$
Simplify.	$\dfrac{4-4\sqrt{3}}{2}$
Factor the common factor from the numerator.	$\dfrac{4(1-\sqrt{3})}{2}$
Remove the common factor, 2, from the numerator and denominator.	$\dfrac{\not{2}\cdot 2(1-\sqrt{3})}{\not{2}}$
Simplify.	$2(1-\sqrt{3})$

TRY IT : : 8.37 Simplify: ⓐ $5+\sqrt{75}$ ⓑ $\dfrac{10-\sqrt{75}}{5}$

TRY IT : : 8.38 Simplify: ⓐ $2+\sqrt{98}$ ⓑ $\dfrac{6-\sqrt{45}}{3}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect power of the index. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

EXAMPLE 8.20

Simplify: ⓐ $\sqrt{\dfrac{45}{80}}$ ⓑ $\sqrt[3]{\dfrac{16}{54}}$ ⓒ $\sqrt[4]{\dfrac{5}{80}}$.

Solution

ⓐ

	$\sqrt{\dfrac{45}{80}}$
Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator.	$\sqrt{\dfrac{5\cdot 9}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt{\dfrac{9}{16}}$
Simplify. Note $\left(\dfrac{3}{4}\right)^2=\dfrac{9}{16}$.	$\dfrac{3}{4}$

ⓑ

	$\sqrt[3]{\frac{16}{54}}$
Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator.	$\sqrt[3]{\frac{2\cdot 8}{2\cdot 27}}$
Simplify the fraction by removing common factors.	$\sqrt[3]{\frac{8}{27}}$
Simplify. Note $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$.	$\frac{2}{3}$

ⓒ

	$\sqrt[4]{\frac{5}{80}}$
Simplify inside the radical fir t. Rewrite showing the common factors of the numerator and denominator.	$\sqrt[4]{\frac{5\cdot 1}{5\cdot 16}}$
Simplify the fraction by removing common factors.	$\sqrt[4]{\frac{1}{16}}$
Simplify. Note $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.	$\frac{1}{2}$

TRY IT : : 8.39 Simplify: ⓐ $\sqrt{\frac{75}{48}}$ ⓑ $\sqrt[3]{\frac{54}{250}}$ ⓒ $\sqrt[4]{\frac{32}{162}}$.

TRY IT : : 8.40 Simplify: ⓐ $\sqrt{\frac{98}{162}}$ ⓑ $\sqrt[3]{\frac{24}{375}}$ ⓒ $\sqrt[4]{\frac{4}{324}}$.

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the Quotient Property to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

EXAMPLE 8.21

Simplify: ⓐ $\sqrt{\frac{m^6}{m^4}}$ ⓑ $\sqrt[3]{\frac{a^8}{a^5}}$ ⓒ $\sqrt[4]{\frac{a^{10}}{a^2}}$.

Solution

ⓐ

	$\sqrt{\frac{m^6}{m^4}}$
Simplify the fraction inside the radical fir t. Divide the like bases by subtracting the exponents.	$\sqrt{m^2}$
Simplify.	$\lvert m\rvert$

ⓑ

	$\sqrt[3]{\frac{a^8}{a^5}}$
Use the Quotient Property of exponents to simplify the fraction under the radical fir t.	$\sqrt[3]{a^3}$
Simplify.	a

ⓒ

	$\sqrt[4]{\frac{a^{10}}{a^2}}$
Use the Quotient Property of exponents to simplify the fraction under the radical fir t.	$\sqrt[4]{a^8}$
Rewrite the radicand using perfect fourth power factors.	$\sqrt[4]{\left(a^2\right)^4}$
Simplify.	a^2

TRY IT : : 8.41 Simplify: ⓐ $\sqrt{\frac{a^8}{a^6}}$ ⓑ $\sqrt[4]{\frac{x^7}{x^3}}$ ⓒ $\sqrt[4]{\frac{y^{17}}{y^5}}$.

TRY IT : : 8.42 Simplify: ⓐ $\sqrt{\frac{x^{14}}{x^{10}}}$ ⓑ $\sqrt[3]{\frac{m^{13}}{m^7}}$ ⓒ $\sqrt[5]{\frac{n^{12}}{n^2}}$.

Remember the Quotient to a Power Property? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$$

We can use a similar property to simplify a root of a fraction. After removing all common factors from the numerator and denominator, if the fraction is not a perfect power of the index, we simplify the numerator and denominator separately.

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

EXAMPLE 8.22 HOW TO SIMPLIFY THE QUOTIENT OF RADICAL EXPRESSIONS

Simplify: $\sqrt{\frac{27m^3}{196}}$.

Solution

Step 1. Simplify the fraction in the radicand, if possible.	$\frac{27m^3}{196}$ cannot be simplified.	$\sqrt{\frac{27m^3}{196}}$

Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.	We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.	$\frac{\sqrt{27m^3}}{\sqrt{196}}$
Step 3. Simplify the radicals in the numerator and the denominator.	$9m^2$ and 196 are perfect squares.	$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$ $\frac{3m\sqrt{3m}}{14}$

TRY IT : : 8.43 Simplify: $\sqrt{\frac{24p^3}{49}}$.

TRY IT : : 8.44 Simplify: $\sqrt{\frac{48x^5}{100}}$.

HOW TO : : SIMPLIFY A SQUARE ROOT USING THE QUOTIENT PROPERTY.

Step 1. Simplify the fraction in the radicand, if possible.
Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
Step 3. Simplify the radicals in the numerator and the denominator.

EXAMPLE 8.23

Simplify: ⓐ $\sqrt{\frac{45x^5}{y^4}}$ ⓑ $\sqrt[3]{\frac{24x^7}{y^3}}$ ⓒ $\sqrt[4]{\frac{48x^{10}}{y^8}}$.

Solution

ⓐ

	$\sqrt{\frac{45x^5}{y^4}}$
We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.	$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt{9x^4} \cdot \sqrt{5x}}{y^2}$
Simplify.	$\frac{3x^2\sqrt{5x}}{y^2}$

ⓑ

$$\sqrt[3]{\frac{24x^7}{y^3}}$$

The fraction in the radicand cannot be simplified. Use he Quotient Property to write as two radicals.

$$\frac{\sqrt[3]{24x^7}}{\sqrt[3]{y^3}}$$

Rewrite each radicand as a product using perfect cube factors.

$$\frac{\sqrt[3]{8x^6 \cdot 3x}}{\sqrt[3]{y^3}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[3]{(2x^2)^3} \cdot \sqrt[3]{3x}}{\sqrt[3]{y^3}}$$

Simplify.

$$\frac{2x^2\sqrt[3]{3x}}{y}$$

ⓒ

$$\sqrt[4]{\frac{48x^{10}}{y^8}}$$

The fraction in the radicand cannot be simplified

$$\frac{\sqrt[4]{48x^{10}}}{\sqrt[4]{y^8}}$$

Use the Quotient Property to write as two radicals. Rewrite each radicand as a product using perfect fourth power factors.

$$\frac{\sqrt[4]{16x^8 \cdot 3x^2}}{\sqrt[4]{y^8}}$$

Rewrite the numerator as the product of two radicals.

$$\frac{\sqrt[4]{(2x^2)^4} \cdot \sqrt[4]{3x^2}}{\sqrt[4]{(y^2)^4}}$$

Simplify.

$$\frac{2x^2\sqrt[4]{3x^2}}{y^2}$$

TRY IT : : 8.45

Simplify: ⓐ $\sqrt{\frac{80m^3}{n^6}}$ ⓑ $\sqrt[3]{\frac{108c^{10}}{d^6}}$ ⓒ $\sqrt[4]{\frac{80x^{10}}{y^4}}$.

TRY IT : : 8.46

Simplify: ⓐ $\sqrt{\frac{54u^7}{v^8}}$ ⓑ $\sqrt[3]{\frac{40r^3}{s^6}}$ ⓒ $\sqrt[4]{\frac{162m^{14}}{n^{12}}}$.

Be sure to simplify the fraction in the radicand first, if possible.

EXAMPLE 8.24

Simplify: ⓐ $\sqrt{\frac{18p^5q^7}{32pq^2}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

Solution

ⓐ

	$\sqrt{\dfrac{18p^5q^7}{32pq^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt{\dfrac{9p^4q^5}{16}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt{9p^4q^5}}{\sqrt{16}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$
Simplify.	$\dfrac{3p^2q^2\sqrt{q}}{4}$

ⓑ

	$\sqrt[3]{\dfrac{16x^5y^7}{54x^2y^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt[3]{\dfrac{8x^3y^5}{27}}$
Rewrite using the Quotient Property.	$\dfrac{\sqrt[3]{8x^3y^5}}{\sqrt[3]{27}}$
Simplify the radicals in the numerator and the denominator.	$\dfrac{\sqrt[3]{8x^3y^3}\cdot\sqrt[3]{y^2}}{\sqrt[3]{27}}$
Simplify.	$\dfrac{2xy\sqrt[3]{y^2}}{3}$

ⓒ

	$\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$
Simplify the fraction in the radicand, if possible.	$\sqrt[4]{\frac{a^5b^4}{16}}$
Rewrite using the Quotient Property.	$\frac{\sqrt[4]{a^5b^4}}{\sqrt[4]{16}}$
Simplify the radicals in the numerator and the denominator.	$\frac{\sqrt[4]{a^4b^4}\cdot\sqrt[4]{a}}{\sqrt[4]{16}}$
Simplify.	$\frac{\lvert ab\rvert\sqrt[4]{a}}{2}$

TRY IT : : 8.47

Simplify: ⓐ $\sqrt{\frac{50x^5y^3}{72x^4y}}$ ⓑ $\sqrt[3]{\frac{16x^5y^7}{54x^2y^2}}$ ⓒ $\sqrt[4]{\frac{5a^8b^6}{80a^3b^2}}$.

TRY IT : : 8.48

Simplify: ⓐ $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$ ⓑ $\sqrt[3]{\frac{54x^7y^5}{250x^2y^2}}$ ⓒ $\sqrt[4]{\frac{32a^9b^7}{162a^3b^3}}$.

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the Quotient Property again, to combine them into one radical. We will then look to see if we can simplify the expression.

EXAMPLE 8.25

Simplify: ⓐ $\frac{\sqrt{48a^7}}{\sqrt{3a}}$ ⓑ $\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$ ⓒ $\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$.

Solution

ⓐ

	$\frac{\sqrt{48a^7}}{\sqrt{3a}}$
The denominator cannot be simplified, s use the Quotient Property to write as one radical.	$\sqrt{\frac{48a^7}{3a}}$
Simplify the fraction under the radical.	$\sqrt{16a^6}$
Simplify.	$4\lvert a^3\rvert$

ⓑ

	$\frac{\sqrt[3]{-108}}{\sqrt[3]{2}}$
The denominator cannot be simplified, s use the Quotient Property to write as one radical.	$\sqrt[3]{\frac{-108}{2}}$
Simplify the fraction under the radical.	$\sqrt[3]{-54}$
Rewrite the radicand as a product using perfect cube factors.	$\sqrt[3]{(-3)^3 \cdot 2}$
Rewrite the radical as the product of two radicals.	$\sqrt[3]{(-3)^3} \cdot \sqrt[3]{2}$
Simplify.	$-3\sqrt[3]{2}$

ⓒ

	$\frac{\sqrt[4]{96x^7}}{\sqrt[4]{3x^2}}$
The denominator cannot be simplified, s use the Quotient Property to write as one radical.	$\sqrt[4]{\frac{96x^7}{3x^2}}$
Simplify the fraction under the radical.	$\sqrt[4]{32x^5}$
Rewrite the radicand as a product using perfect fourth power factors.	$\sqrt[4]{16x^4} \cdot \sqrt[4]{2x}$
Rewrite the radical as the product of two radicals.	$\sqrt[4]{(2x)^4} \cdot \sqrt[4]{2x}$
Simplify.	$2\lvert x\rvert\sqrt[4]{2x}$

TRY IT : : 8.49

Simplify: ⓐ $\frac{\sqrt{98z^5}}{\sqrt{2z}}$ ⓑ $\frac{\sqrt[3]{-500}}{\sqrt[3]{2}}$ ⓒ $\frac{\sqrt[4]{486m^{11}}}{\sqrt[4]{3m^5}}$.

TRY IT : : 8.50

Simplify: ⓐ $\frac{\sqrt{128m^9}}{\sqrt{2m}}$ ⓑ $\frac{\sqrt[3]{-192}}{\sqrt[3]{3}}$ ⓒ $\frac{\sqrt[4]{324n^7}}{\sqrt[4]{2n^3}}$.

MEDIA : :

Access these online resources for additional instruction and practice with simplifying radical expressions.

- **Simplifying Square Root and Cube Root with Variables (https://openstax.org/l/37SimRtwithVar1)**
- **Express a Radical in Simplified Form-Square and Cube Roots with Variables and Exponents (https://openstax.org/l/37SimRtwithVar2)**
- **Simplifying Cube Roots (https://openstax.org/l/37SimRtwithVar3)**

8.2 EXERCISES

Practice Makes Perfect

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

55. $\sqrt{27}$

56. $\sqrt{80}$

57. $\sqrt{125}$

58. $\sqrt{96}$

59. $\sqrt{147}$

60. $\sqrt{450}$

61. $\sqrt{800}$

62. $\sqrt{675}$

63. ⓐ $\sqrt[4]{32}$ ⓑ $\sqrt[5]{64}$

64. ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

65. ⓐ $\sqrt[5]{64}$ ⓑ $\sqrt[3]{256}$

66. ⓐ $\sqrt[4]{3125}$ ⓑ $\sqrt[3]{81}$

In the following exercises, simplify using absolute value signs as needed.

67.
ⓐ $\sqrt{y^{11}}$
ⓑ $\sqrt[3]{r^5}$
ⓒ $\sqrt[4]{s^{10}}$

68.
ⓐ $\sqrt{m^{13}}$
ⓑ $\sqrt[5]{u^7}$
ⓒ $\sqrt[6]{v^{11}}$

69.
ⓐ $\sqrt{n^{21}}$
ⓑ $\sqrt[3]{q^8}$
ⓒ $\sqrt[8]{n^{10}}$

70.
ⓐ $\sqrt{r^{25}}$
ⓑ $\sqrt[5]{p^8}$
ⓒ $\sqrt[4]{m^5}$

71.
ⓐ $\sqrt{125r^{13}}$
ⓑ $\sqrt[3]{108x^5}$
ⓒ $\sqrt[4]{48y^6}$

72.
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

73.
ⓐ $\sqrt{242m^{23}}$
ⓑ $\sqrt[4]{405m10}$
ⓒ $\sqrt[5]{160n^8}$

74.
ⓐ $\sqrt{175n^{13}}$
ⓑ $\sqrt[5]{512p^5}$
ⓒ $\sqrt[4]{324q^7}$

75.
ⓐ $\sqrt{147m^7n^{11}}$
ⓑ $\sqrt[3]{48x^6y^7}$
ⓒ $\sqrt[4]{32x^5y^4}$

76.
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

77.
ⓐ $\sqrt{192q^3r^7}$
ⓑ $\sqrt[3]{54m^9n^{10}}$
ⓒ $\sqrt[4]{81a^9b^8}$

78.
ⓐ $\sqrt{150m^9n^3}$
ⓑ $\sqrt[3]{81p^7q^8}$
ⓒ $\sqrt[4]{162c^{11}d^{12}}$

79.
ⓐ $\sqrt[3]{-864}$
ⓑ $\sqrt[4]{-256}$

80.
ⓐ $\sqrt[5]{-486}$
ⓑ $\sqrt[6]{-64}$

81.
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

82.
ⓐ $\sqrt[3]{-8}$
ⓑ $\sqrt[4]{-16}$

83.
ⓐ $5+\sqrt{12}$
ⓑ $\dfrac{10-\sqrt{24}}{2}$

84.
ⓐ $8+\sqrt{96}$
ⓑ $\dfrac{8-\sqrt{80}}{4}$

85.
ⓐ $1+\sqrt{45}$
ⓑ $\dfrac{3+\sqrt{90}}{3}$

86.
ⓐ $3+\sqrt{125}$
ⓑ $\dfrac{15+\sqrt{75}}{5}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

87. ⓐ $\sqrt{\dfrac{45}{80}}$ ⓑ $\sqrt[3]{\dfrac{8}{27}}$ ⓒ $\sqrt[4]{\dfrac{1}{81}}$

88. ⓐ $\sqrt{\dfrac{72}{98}}$ ⓑ $\sqrt[3]{\dfrac{24}{81}}$ ⓒ $\sqrt[4]{\dfrac{6}{96}}$

89. ⓐ $\sqrt{\dfrac{100}{36}}$ ⓑ $\sqrt[3]{\dfrac{81}{375}}$ ⓒ $\sqrt[4]{\dfrac{1}{256}}$

90. ⓐ $\sqrt{\dfrac{121}{16}}$ ⓑ $\sqrt[3]{\dfrac{16}{250}}$ ⓒ $\sqrt[4]{\dfrac{32}{162}}$

91. ⓐ $\sqrt{\dfrac{x^{10}}{x^6}}$ ⓑ $\sqrt[3]{\dfrac{p^{11}}{p^2}}$ ⓒ $\sqrt[4]{\dfrac{q^{17}}{q^{13}}}$

92. ⓐ $\sqrt{\dfrac{p^{20}}{p^{10}}}$ ⓑ $\sqrt[5]{\dfrac{d^{12}}{d^7}}$ ⓒ $\sqrt[8]{\dfrac{m^{12}}{m^4}}$

93. ⓐ $\sqrt{\dfrac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\dfrac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\dfrac{v^{30}}{v^{12}}}$

94. ⓐ $\sqrt{\dfrac{q^8}{q^{14}}}$ ⓑ $\sqrt[3]{\dfrac{r^{14}}{r^5}}$ ⓒ $\sqrt[4]{\dfrac{c^{21}}{c^9}}$

95. $\sqrt{\dfrac{96x^7}{121}}$

96. $\sqrt{\dfrac{108y^4}{49}}$

97. $\sqrt{\dfrac{300m^5}{64}}$

98. $\sqrt{\dfrac{125n^7}{169}}$

99. $\sqrt{\dfrac{98r^5}{100}}$

100. $\sqrt{\dfrac{180s^{10}}{144}}$

101. $\sqrt{\dfrac{28q^6}{225}}$

102. $\sqrt{\dfrac{150r^3}{256}}$

103.
ⓐ $\sqrt{\dfrac{75r^9}{s^8}}$
ⓑ $\sqrt[3]{\dfrac{54a^8}{b^3}}$
ⓒ $\sqrt[4]{\dfrac{64c^5}{d^4}}$

104.
ⓐ $\sqrt{\dfrac{72x^5}{y^6}}$
ⓑ $\sqrt[5]{\dfrac{96r^{11}}{s^5}}$
ⓒ $\sqrt[6]{\dfrac{128u^7}{v^{12}}}$

105.
ⓐ $\sqrt{\dfrac{28p^7}{q^2}}$
ⓑ $\sqrt[3]{\dfrac{81s^8}{t^3}}$
ⓒ $\sqrt[4]{\dfrac{64p^{15}}{q^{12}}}$

106.
ⓐ $\sqrt{\dfrac{45r^3}{s^{10}}}$
ⓑ $\sqrt[3]{\dfrac{625u^{10}}{v^3}}$
ⓒ $\sqrt[4]{\dfrac{729c^{21}}{d^8}}$

107.
ⓐ $\sqrt{\dfrac{32x^5y^3}{18x^3y}}$
ⓑ $\sqrt[3]{\dfrac{5x^6y^9}{40x^5y^3}}$
ⓒ $\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$

108.
ⓐ $\sqrt{\dfrac{75r^6s^8}{48rs^4}}$
ⓑ $\sqrt[3]{\dfrac{24x^8y^4}{81x^2y}}$
ⓒ $\sqrt[4]{\dfrac{32m^9n^2}{162mn^2}}$

109.
ⓐ $\sqrt{\dfrac{27p^2q}{108p^4q^3}}$
ⓑ $\sqrt[3]{\dfrac{16c^5d^7}{250c^2d^2}}$
ⓒ $\sqrt[6]{\dfrac{2m^9n^7}{128m^3n}}$

110.
ⓐ $\sqrt{\dfrac{50r^5s^2}{128r^2s^6}}$
ⓑ $\sqrt[3]{\dfrac{24m^9n^7}{375m^4n}}$
ⓒ $\sqrt[4]{\dfrac{81m^2n^8}{256m^1n^2}}$

111.

ⓐ $\dfrac{\sqrt{45p^9}}{\sqrt{5q^2}}$

ⓑ $\dfrac{\sqrt[4]{64}}{\sqrt[4]{2}}$

ⓒ $\dfrac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

112.

ⓐ $\dfrac{\sqrt{80q^5}}{\sqrt{5q}}$

ⓑ $\dfrac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

ⓒ $\dfrac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

113.

ⓐ $\dfrac{\sqrt{50m^7}}{\sqrt{2m}}$

ⓑ $\sqrt[3]{\dfrac{1250}{2}}$

ⓒ $\sqrt[4]{\dfrac{486y^9}{2y^3}}$

114.

ⓐ $\dfrac{\sqrt{72n^{11}}}{\sqrt{2n}}$

ⓑ $\sqrt[3]{\dfrac{162}{6}}$

ⓒ $\sqrt[4]{\dfrac{160r^{10}}{5r^3}}$

Writing Exercises

115. Explain why $\sqrt{x^4} = x^2$. Then explain why $\sqrt{x^{16}} = x^8$.

116. Explain why $7 + \sqrt{9}$ is not equal to $\sqrt{7 + 9}$.

117. Explain how you know that $\sqrt[5]{x^{10}} = x^2$.

118. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

8.3 Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{15}+\frac{5}{12}$.
   If you missed this problem, review **Example 1.28**.
2. Simplify: $\left(4x^2y^5\right)^3$.
   If you missed this problem, review **Example 5.18**.
3. Simplify: 5^{-3}.
   If you missed this problem, review **Example 5.14**.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n = a^{m\cdot n}$ when *m* and *n* are whole numbers. Let's assume we are now not limited to whole numbers.

Suppose we want to find a number *p* such that $(8^p)^3 = 8.$ We will use the Power Property of Exponents to find the value of *p*.

	$(8^p)^3 = 8$
Multiply the exponents on the left.	$8^{3p} = 8$
Write the exponent 1 on the right.	$8^{3p} = 8^1$
Since the bases are the same, the exponents must be equal.	$3p = 1$
Solve for p.	$p = \frac{1}{3}$

So $\left(8^{\frac{1}{3}}\right)^3 = 8.$ But we know also $\left(\sqrt[3]{8}\right)^3 = 8.$ Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}.$

This same logic can be used for any positive integer exponent *n* to show that $a^{\frac{1}{n}} = \sqrt[n]{a}.$

Rational Exponent $a^{\frac{1}{n}}$

If $\sqrt[n]{a}$ is a real number and $n \geq 2,$ then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

EXAMPLE 8.26

Write as a radical expression: ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$.

Solution

We want to write each expression in the form $\sqrt[n]{a}$.

ⓐ

	$x^{\frac{1}{2}}$
The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$

ⓑ

	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$

ⓒ

	$z^{\frac{1}{4}}$
The denominator of the exponent is 4, so the index is 4.	$\sqrt[4]{z}$

TRY IT : : 8.51

Write as a radical expression: ⓐ $t^{\frac{1}{2}}$ ⓑ $m^{\frac{1}{3}}$ ⓒ $r^{\frac{1}{4}}$.

TRY IT : : 8.52

Write as a radial expression: ⓐ $b^{\frac{1}{6}}$ ⓑ $z^{\frac{1}{5}}$ ⓒ $p^{\frac{1}{4}}$.

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

EXAMPLE 8.27

Write with a rational exponent: ⓐ $\sqrt{5y}$ ⓑ $\sqrt[3]{4x}$ ⓒ $3\sqrt[4]{5z}$.

Solution

We want to write each radical in the form $a^{\frac{1}{n}}$.

ⓐ

	$\sqrt{5y}$
No index is shown, so it is 2. The denominator of the exponent will be 2. Put parentheses around the entire expression $5y$.	$(5y)^{\frac{1}{2}}$

ⓑ

	$\sqrt[3]{4x}$
The index is 3, so the denominator of the exponent is 3. Include parentheses $(4x)$.	$(4x)^{\frac{1}{3}}$

ⓒ

	$3\sqrt[4]{5z}$
The index is 4, so the denominator of the exponent is 4. Put parentheses only around the $5z$ since 3 is not under the radical sign.	$3(5z)^{\frac{1}{4}}$

TRY IT : : 8.53 Write with a rational exponent: ⓐ $\sqrt{10m}$ ⓑ $\sqrt[5]{3n}$ ⓒ $3\sqrt[4]{6y}$.

TRY IT : : 8.54 Write with a rational exponent: ⓐ $\sqrt[7]{3k}$ ⓑ $\sqrt[4]{5j}$ ⓒ $8\sqrt[3]{2a}$.

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

EXAMPLE 8.28

Simplify: ⓐ $25^{\frac{1}{2}}$ ⓑ $64^{\frac{1}{3}}$ ⓒ $256^{\frac{1}{4}}$.

Solution

ⓐ

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

ⓑ

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

ⓒ

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

TRY IT : : 8.55 Simplify: ⓐ $36^{\frac{1}{2}}$ ⓑ $8^{\frac{1}{3}}$ ⓒ $16^{\frac{1}{4}}$.

> **TRY IT : :** 8.56
>
> Simplify: ⓐ $100^{\frac{1}{2}}$ ⓑ $27^{\frac{1}{3}}$ ⓒ $81^{\frac{1}{4}}$.

Be careful of the placement of the negative signs in the next example. We will need to use the property $a^{-n} = \frac{1}{a^n}$ in one case.

EXAMPLE 8.29

Simplify: ⓐ $(-16)^{\frac{1}{4}}$ ⓑ $-16^{\frac{1}{4}}$ ⓒ $(16)^{-\frac{1}{4}}$.

Solution

ⓐ

	$(-16)^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{-16}$
	$\sqrt[4]{(-2)^4}$
Simplify.	No real solution.

ⓑ

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fouth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

ⓒ

	$(16)^{-\frac{1}{4}}$
Rewrite using the property $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$
Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

> **TRY IT : :** 8.57
>
> Simplify: ⓐ $(-64)^{-\frac{1}{2}}$ ⓑ $-64^{\frac{1}{2}}$ ⓒ $(64)^{-\frac{1}{2}}$.

> **TRY IT : :** 8.58
>
> Simplify: ⓐ $(-256)^{\frac{1}{4}}$ ⓑ $-256^{\frac{1}{4}}$ ⓒ $(256)^{-\frac{1}{4}}$.

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and

$(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

This leads us to the following definition.

Rational Exponent $a^{\frac{m}{n}}$

For any positive integers *m* and *n*,

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

EXAMPLE 8.30

Write with a rational exponent: ⓐ $\sqrt{y^3}$ ⓑ $\left(\sqrt[3]{2x}\right)^4$ ⓒ $\sqrt{\left(\frac{3a}{4b}\right)^3}$.

Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$.

ⓐ

	$\sqrt{y^3}$
The numerator of the exponent is the exponent, 3.	
The denominator of the exponent is the index of the radical, 2.	$y^{\frac{3}{2}}$

ⓑ

	$\left(\sqrt[3]{2x}\right)^4$
The numerator of the exponent is the exponent, 4.	
The denominator of the exponent is the index of the radical, 3.	$(2x)^{\frac{4}{3}}$

ⓒ

	$\sqrt{\left(\frac{3a}{4b}\right)^3}$
The numerator of the exponent is the exponent, 3.	
The denominator of the exponent is the index of the radical, 2.	$\left(\frac{3a}{4b}\right)^{\frac{3}{2}}$

TRY IT : : 8.59

Write with a rational exponent: ⓐ $\sqrt{x^5}$ ⓑ $\left(\sqrt[4]{3y}\right)^3$ ⓒ $\sqrt{\left(\frac{2m}{3n}\right)^5}$.

TRY IT : : 8.60

Write with a rational exponent: ⓐ $\sqrt[5]{a^2}$ ⓑ $\left(\sqrt[3]{5ab}\right)^5$ ⓒ $\sqrt{\left(\frac{7xy}{z}\right)^3}$.

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

EXAMPLE 8.31

Simplify: ⓐ $125^{\frac{2}{3}}$ ⓑ $16^{-\frac{3}{2}}$ ⓒ $32^{-\frac{2}{5}}$.

Solution

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

ⓐ

	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	$\left(\sqrt[3]{125}\right)^2$
Simplify.	$(5)^2$
	25

ⓑ We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

	$16^{-\frac{3}{2}}$
Rewrite using $a^{-n} = \frac{1}{a^n}$	$\frac{1}{16^{\frac{3}{2}}}$
Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.	$\frac{1}{(\sqrt{16})^3}$
Simplify.	$\frac{1}{4^3}$
	$\frac{1}{64}$

ⓒ

	$32^{-\frac{2}{5}}$
Rewrite using $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{32^{\frac{2}{5}}}$
Change to radical form.	$\frac{1}{\left(\sqrt[5]{32}\right)^2}$
Rewrite the radicand as a power.	$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$
Simplify.	$\frac{1}{2^2}$
	$\frac{1}{4}$

TRY IT : : 8.61

Simplify: ⓐ $27^{\frac{2}{3}}$ ⓑ $81^{-\frac{3}{2}}$ ⓒ $16^{-\frac{3}{4}}$.

TRY IT : : 8.62

Simplify: ⓐ $4^{\frac{3}{2}}$ ⓑ $27^{-\frac{2}{3}}$ ⓒ $625^{-\frac{3}{4}}$.

EXAMPLE 8.32

Simplify: ⓐ $-25^{\frac{3}{2}}$ ⓑ $-25^{-\frac{3}{2}}$ ⓒ $(-25)^{\frac{3}{2}}$.

Solution

ⓐ

	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-\left(\sqrt{25}\right)^3$
Simplify the radical.	$-(5)^3$
Simplify.	-125

ⓑ

	$-25^{-\frac{3}{2}}$
Rewrite using $a^{-n} = \frac{1}{a^n}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{(\sqrt{25})^3}\right)$
Simplify the radical.	$-\left(\frac{1}{(5)^3}\right)$
Simplify.	$-\frac{1}{125}$

ⓒ

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$(\sqrt{-25})^3$
There is no real number whose square root is -25.	Not a real number.

TRY IT : : 8.63

Simplify: ⓐ $-16^{\frac{3}{2}}$ ⓑ $-16^{-\frac{3}{2}}$ ⓒ $(-16)^{-\frac{3}{2}}$.

TRY IT : : 8.64

Simplify: ⓐ $-81^{\frac{3}{2}}$ ⓑ $-81^{-\frac{3}{2}}$ ⓒ $(-81)^{-\frac{3}{2}}$.

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used also apply to rational exponents. We will list the Properties of Exponenets here to have them for reference as we simplify expressions.

Properties of Exponents

If *a* and *b* are real numbers and *m* and *n* are rational numbers, then

Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^m = a^m b^m$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, \ a \neq 0$
Zero Exponent Definitio	$a^0 = 1, \ a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \ b \neq 0$
Negative Exponent Property	$a^{-n} = \frac{1}{a^n}, \ a \neq 0$

We will apply these properties in the next example.

EXAMPLE 8.33

Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$ ⓑ $\left(z^9\right)^{\frac{2}{3}}$ ⓒ $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$.

Solution

ⓐ The Product Property tells us that when we multiply the same base, we add the exponents.

	$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$
The bases are the same, so we add the exponents.	$x^{\frac{1}{2}+\frac{5}{6}}$
Add the fractions.	$x^{\frac{8}{6}}$
Simplify the exponent.	$x^{\frac{4}{3}}$

ⓑ The Power Property tells us that when we raise a power to a power, we multiply the exponents.

	$\left(z^9\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$z^{9 \cdot \frac{2}{3}}$
Simplify.	z^6

ⓒ The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

	$\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
	$\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$
To divide with the same base, we subtract the exponents.	$\dfrac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$
Simplify.	$\dfrac{1}{x^{\frac{4}{3}}}$

TRY IT : : 8.65

Simplify: ⓐ $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$ ⓑ $\left(x^6\right)^{\frac{4}{3}}$ ⓒ $\dfrac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$.

TRY IT : : 8.66

Simplify: ⓐ $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$ ⓑ $\left(m^9\right)^{\frac{2}{9}}$ ⓒ $\dfrac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$.

Sometimes we need to use more than one property. In the next example, we will use both the Product to a Power Property and then the Power Property.

EXAMPLE 8.34

Simplify: ⓐ $\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$ ⓑ $\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$.

Solution

ⓐ

	$\left(27u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
First we use the Product to a Power Property.	$(27)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
Rewrite 27 as a power of 3.	$\left(3^3\right)^{\frac{2}{3}}\left(u^{\frac{1}{2}}\right)^{\frac{2}{3}}$
To raise a power to a power, we multiply the exponents.	$\left(3^2\right)\left(u^{\frac{1}{3}}\right)$
Simplify.	$9u^{\frac{1}{3}}$

ⓑ

	$\left(m^{\frac{2}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{2}}$
First we use the Product to a Power Property.	$\left(m^{\frac{2}{3}}\right)^{\frac{3}{2}}\left(n^{\frac{1}{2}}\right)^{\frac{3}{2}}$
To raise a power to a power, we multiply the exponents.	$mn^{\frac{3}{4}}$

TRY IT : : 8.67

Simplify: ⓐ $\left(32x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ ⓑ $\left(x^{\frac{3}{4}}y^{\frac{1}{2}}\right)^{\frac{2}{3}}$.

TRY IT : : 8.68

Simplify: ⓐ $\left(81n^{\frac{2}{5}}\right)^{\frac{3}{2}}$ ⓑ $\left(a^{\frac{3}{2}}b^{\frac{1}{2}}\right)^{\frac{4}{3}}$.

We will use both the Product Property and the Quotient Property in the next example.

EXAMPLE 8.35

Simplify: ⓐ $\dfrac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$ ⓑ $\left(\dfrac{16\,x^{\frac{4}{3}}\,y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}\,y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$.

Solution

ⓐ

	$\dfrac{x^{\frac{3}{4}} \cdot x^{-\frac{1}{4}}}{x^{-\frac{6}{4}}}$
Use the Product Property in the numerator, add the exponents.	$\dfrac{x^{\frac{2}{4}}}{x^{-\frac{6}{4}}}$
Use the Quotient Property, subtract the exponents.	$x^{\frac{8}{4}}$
Simplify.	x^2

ⓑ Follow the order of operations to simplify inside the parenthese first.

	$\left(\dfrac{16\,x^{\frac{4}{3}}\,y^{-\frac{5}{6}}}{x^{-\frac{2}{3}}\,y^{\frac{1}{6}}}\right)^{\frac{1}{2}}$
Use the Quotient Property, subtract the exponents.	$\left(\dfrac{16x^{\frac{6}{3}}}{y^{\frac{6}{6}}}\right)^{\frac{1}{2}}$
Simplify.	$\left(\dfrac{16x^2}{y}\right)^{\frac{1}{2}}$
Use the Product to a Power Property, multiply the exponents.	$\dfrac{4x}{y^{\frac{1}{2}}}$

TRY IT : : 8.69

Simplify: ⓐ $\dfrac{m^{\frac{2}{3}} \cdot m^{-\frac{1}{3}}}{m^{-\frac{5}{3}}}$ ⓑ $\left(\dfrac{25m^{\frac{1}{6}}\,n^{\frac{11}{6}}}{m^{\frac{2}{3}}\,n^{-\frac{1}{6}}}\right)^{\frac{1}{2}}$.

TRY IT : : 8.70

Simplify: ⓐ $\dfrac{u^{\frac{4}{5}} \cdot u^{-\frac{2}{5}}}{u^{-\frac{13}{5}}}$ ⓑ $\left(\dfrac{27x^{\frac{4}{5}}\,y^{\frac{1}{6}}}{x^{\frac{1}{5}}\,y^{-\frac{5}{6}}}\right)^{\frac{1}{3}}$.

MEDIA : :

Access these online resources for additional instruction and practice with simplifying rational exponents.

- **Review-Rational Exponents (https://openstax.org/l/37RatExpont1)**
- **Using Laws of Exponents on Radicals: Properties of Rational Exponents (https://openstax.org/l/37RatExpont2)**

8.3 EXERCISES

Practice Makes Perfect

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

119. ⓐ $x^{\frac{1}{2}}$ ⓑ $y^{\frac{1}{3}}$ ⓒ $z^{\frac{1}{4}}$

120. ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

121. ⓐ $u^{\frac{1}{5}}$ ⓑ $v^{\frac{1}{9}}$ ⓒ $w^{\frac{1}{20}}$

122. ⓐ $g^{\frac{1}{7}}$ ⓑ $h^{\frac{1}{5}}$ ⓒ $j^{\frac{1}{25}}$

In the following exercises, write with a rational exponent.

123. ⓐ $\sqrt[7]{x}$ ⓑ $\sqrt[9]{y}$ ⓒ $\sqrt[5]{f}$

124. ⓐ $\sqrt[8]{r}$ ⓑ $\sqrt[10]{s}$ ⓒ $\sqrt[4]{t}$

125. ⓐ $\sqrt[3]{7c}$ ⓑ $\sqrt[7]{12d}$ ⓒ $2\sqrt[4]{6b}$

126. ⓐ $\sqrt[4]{5x}$ ⓑ $\sqrt[8]{9y}$ ⓒ $7\sqrt[5]{3z}$

127. ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

128. ⓐ $\sqrt[3]{25a}$ ⓑ $\sqrt{3b}$ ⓒ $\sqrt[8]{40c}$

In the following exercises, simplify.

129.
ⓐ $81^{\frac{1}{2}}$
ⓑ $125^{\frac{1}{3}}$
ⓒ $64^{\frac{1}{2}}$

130.
ⓐ $625^{\frac{1}{4}}$
ⓑ $243^{\frac{1}{5}}$
ⓒ $32^{\frac{1}{5}}$

131.
ⓐ $16^{\frac{1}{4}}$
ⓑ $16^{\frac{1}{2}}$
ⓒ $625^{\frac{1}{4}}$

132.
ⓐ $64^{\frac{1}{3}}$
ⓑ $32^{\frac{1}{5}}$
ⓒ $81^{\frac{1}{4}}$

133.
ⓐ $(-216)^{\frac{1}{3}}$
ⓑ $-216^{\frac{1}{3}}$
ⓒ $(216)^{-\frac{1}{3}}$

134.
ⓐ $(-1000)^{\frac{1}{3}}$
ⓑ $-1000^{\frac{1}{3}}$
ⓒ $(1000)^{-\frac{1}{3}}$

135.
ⓐ $(-81)^{\frac{1}{4}}$
ⓑ $-81^{\frac{1}{4}}$
ⓒ $(81)^{-\frac{1}{4}}$

136.
ⓐ $(-49)^{\frac{1}{2}}$
ⓑ $-49^{\frac{1}{2}}$
ⓒ $(49)^{-\frac{1}{2}}$

137.
ⓐ $(-36)^{\frac{1}{2}}$
ⓑ $-36^{\frac{1}{2}}$
ⓒ $(36)^{-\frac{1}{2}}$

138.
ⓐ $(-16)^{\frac{1}{4}}$
ⓑ $-16^{\frac{1}{4}}$
ⓒ $16^{-\frac{1}{4}}$

139.
ⓐ $(-100)^{\frac{1}{2}}$
ⓑ $-100^{\frac{1}{2}}$
ⓒ $(100)^{-\frac{1}{2}}$

140.
ⓐ $(-32)^{\frac{1}{5}}$
ⓑ $(243)^{-\frac{1}{5}}$
ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

141.
ⓐ $\sqrt{m^5}$
ⓑ $\left(\sqrt[3]{3y}\right)^7$
ⓒ $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$

142.
ⓐ $\sqrt[4]{r^7}$
ⓑ $\left(\sqrt[5]{2pq}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

143.
ⓐ $\sqrt[5]{u^2}$
ⓑ $\left(\sqrt[3]{6x}\right)^5$
ⓒ $\sqrt[4]{\left(\frac{18a}{5b}\right)^7}$

144.
ⓐ $\sqrt[3]{a}$
ⓑ $\left(\sqrt[4]{21v}\right)^3$
ⓒ $\sqrt[4]{\left(\frac{2xy}{5z}\right)^2}$

In the following exercises, simplify.

145.
ⓐ $64^{\frac{5}{2}}$
ⓑ $81^{\frac{-3}{2}}$
ⓒ $(-27)^{\frac{2}{3}}$

146.
ⓐ $25^{\frac{3}{2}}$
ⓑ $9^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{2}{3}}$

147.
ⓐ $32^{\frac{2}{5}}$
ⓑ $27^{-\frac{2}{3}}$
ⓒ $(-25)^{\frac{1}{2}}$

148.
ⓐ $100^{\frac{3}{2}}$
ⓑ $49^{-\frac{5}{2}}$
ⓒ $(-100)^{\frac{3}{2}}$

149.
ⓐ $-9^{\frac{3}{2}}$
ⓑ $-9^{-\frac{3}{2}}$
ⓒ $(-9)^{\frac{3}{2}}$

150.
ⓐ $-64^{\frac{3}{2}}$
ⓑ $-64^{-\frac{3}{2}}$
ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

151.
ⓐ $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$
ⓑ $\left(p^{12}\right)^{\frac{3}{4}}$
ⓒ $\frac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$

152.
ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$
ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

153.
ⓐ $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$
ⓑ $\left(x^{12}\right)^{\frac{2}{3}}$
ⓒ $\frac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$

154.
ⓐ $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$
ⓑ $\left(h^6\right)^{\frac{4}{3}}$
ⓒ $\frac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$

155.
ⓐ $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$
ⓑ $\left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

156.
ⓐ $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$
ⓑ $\left(m^{\frac{4}{3}} n^{\frac{1}{2}}\right)^{\frac{3}{4}}$

157.
ⓐ $\left(16\,u^{\frac{1}{3}}\right)^{\frac{3}{4}}$
ⓑ $\left(4\,p^{\frac{1}{3}} q^{\frac{1}{2}}\right)^{\frac{3}{2}}$

158.
ⓐ $\left(625\,n^{\frac{8}{3}}\right)^{\frac{3}{4}}$
ⓑ $\left(9\,x^{\frac{2}{5}} y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

159.
ⓐ $\frac{r^{\frac{5}{2}} \cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$
ⓑ $\left(\frac{36\,s^{\frac{1}{5}} t^{-\frac{3}{2}}}{s^{-\frac{9}{5}} t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

160.
ⓐ $\frac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$
ⓑ $\left(\frac{27\,b^{\frac{2}{3}} c^{-\frac{5}{2}}}{b^{-\frac{7}{3}} c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

161.
ⓐ $\frac{c^{\frac{5}{3}} \cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$
ⓑ $\left(\frac{8\,x^{\frac{5}{3}} y^{-\frac{1}{2}}}{27\,x^{-\frac{4}{3}} y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

162.
ⓐ $\frac{m^{\frac{7}{4}} \cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$
ⓑ $\left(\frac{16\,m^{\frac{1}{5}} n^{\frac{3}{2}}}{81\,m^{\frac{9}{5}} n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Writing Exercises

163. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

164. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

8.4 Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Be Prepared!

Before you get started, take this readiness quiz.

1. Add: $3x^2 + 9x - 5 - \left(x^2 - 2x + 3\right)$.
   If you missed this problem, review **Example 5.5**.
2. Simplify: $(2 + a)(4 - a)$.
   If you missed this problem, review **Example 5.28**.
3. Simplify: $(9 - 5y)^2$.
   If you missed this problem, review **Example 5.31**.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x + 8x$ is $11x$. Similarly we add $3\sqrt{x} + 8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

EXAMPLE 8.36

Simplify: ⓐ $2\sqrt{2} - 7\sqrt{2}$ ⓑ $5\sqrt[3]{y} + 4\sqrt[3]{y}$ ⓒ $7\sqrt[4]{x} - 2\sqrt[4]{y}$.

Solution

ⓐ

	$2\sqrt{2} - 7\sqrt{2}$
Since the radicals are like, we subtract the coefficient	$-5\sqrt{2}$

ⓑ

	$5\sqrt[3]{y} + 4\sqrt[3]{y}$
Since the radicals are like, we add the coefficient	$9\sqrt[3]{y}$

ⓒ

$$7\sqrt[4]{x} - 2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

TRY IT : : 8.71 Simplify: ⓐ $8\sqrt{2} - 9\sqrt{2}$ ⓑ $4\sqrt[3]{x} + 7\sqrt[3]{x}$ ⓒ $3\sqrt[4]{x} - 5\sqrt[4]{y}$.

TRY IT : : 8.72 Simplify: ⓐ $5\sqrt{3} - 9\sqrt{3}$ ⓑ $5\sqrt[3]{y} + 3\sqrt[3]{y}$ ⓒ $5\sqrt[4]{m} - 2\sqrt[3]{m}$.

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

EXAMPLE 8.37

Simplify: ⓐ $2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$ ⓑ $\sqrt[4]{3xy} + 5\sqrt[4]{3xy} - 4\sqrt[4]{3xy}$.

Solution

ⓐ

	$2\sqrt{5n} - 6\sqrt{5n} + 4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

ⓑ

	$\sqrt[4]{3xy} + 5\sqrt[4]{3xy} - 4\sqrt[4]{3xy}$
Since the radicals are like, we combine them.	$2\sqrt[4]{3xy}$

TRY IT : : 8.73 Simplify: ⓐ $\sqrt{7x} - 7\sqrt{7x} + 4\sqrt{7x}$ ⓑ $4\sqrt[4]{5xy} + 2\sqrt[4]{5xy} - 7\sqrt[4]{5xy}$.

TRY IT : : 8.74 Simplify: ⓐ $4\sqrt{3y} - 7\sqrt{3y} + 2\sqrt{3y}$ ⓑ $6\sqrt[3]{7mn} + \sqrt[3]{7mn} - 4\sqrt[3]{7mn}$.

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

EXAMPLE 8.38

Simplify: ⓐ $\sqrt{20} + 3\sqrt{5}$ ⓑ $\sqrt[3]{24} - \sqrt[3]{375}$ ⓒ $\frac{1}{2}\sqrt[4]{48} - \frac{2}{3}\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{20} + 3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4} \cdot \sqrt{5} + 3\sqrt{5}$
	$2\sqrt{5} + 3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

ⓑ

	$\sqrt[3]{24} - \sqrt[3]{375}$
Simplify the radicals.	$\sqrt[3]{8} \cdot \sqrt[3]{3} - \sqrt[3]{125} \cdot \sqrt[3]{3}$
	$2\sqrt[3]{3} - 5\sqrt[3]{3}$
Combine the like radicals.	$-3\sqrt[3]{3}$

ⓒ

8.4 Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

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Before you get started, take this readiness quiz.

1. Add: $3x^2 + 9x - 5 - \left(x^2 - 2x + 3\right)$.
   If you missed this problem, review Example 5.5.
2. Simplify: $(2 + a)(4 - a)$.
   If you missed this problem, review Example 5.28.
3. Simplify: $(9 - 5y)^2$.
   If you missed this problem, review Example 5.31.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Like Radicals

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x + 8x$ is $11x$. Similarly we add $3\sqrt{x} + 8\sqrt{x}$ and the result is $11\sqrt{x}$.

Think about adding like terms with variables as you do the next few examples. When you have like radicals, you just add or subtract the coefficients. When the radicals are not like, you cannot combine the terms.

EXAMPLE 8.36

Simplify: ⓐ $2\sqrt{2} - 7\sqrt{2}$ ⓑ $5\sqrt[3]{y} + 4\sqrt[3]{y}$ ⓒ $7\sqrt[4]{x} - 2\sqrt[4]{y}$.

Solution

ⓐ

	$2\sqrt{2} - 7\sqrt{2}$
Since the radicals are like, we subtract the coefficient	$-5\sqrt{2}$

ⓑ

	$5\sqrt[3]{y} + 4\sqrt[3]{y}$
Since the radicals are like, we add the coefficient	$9\sqrt[3]{y}$

ⓒ

$$7\sqrt[4]{x} - 2\sqrt[4]{y}$$

The indices are the same but the radicals are different. These are not like radicals. Since the radicals are not like, we cannot subtract them.

TRY IT : : 8.71 Simplify: ⓐ $8\sqrt{2} - 9\sqrt{2}$ ⓑ $4\sqrt[3]{x} + 7\sqrt[3]{x}$ ⓒ $3\sqrt[4]{x} - 5\sqrt[4]{y}$.

TRY IT : : 8.72 Simplify: ⓐ $5\sqrt{3}-9\sqrt{3}$ ⓑ $5\sqrt[3]{y}+3\sqrt[3]{y}$ ⓒ $5\sqrt[4]{m}-2\sqrt[3]{m}$.

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

EXAMPLE 8.37

Simplify: ⓐ $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$ ⓑ $\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$.

Solution

ⓐ

	$2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$
Since the radicals are like, we combine them.	$0\sqrt{5n}$
Simplify.	0

ⓑ

	$\sqrt[4]{3xy}+5\sqrt[4]{3xy}-4\sqrt[4]{3xy}$
Since the radicals are like, we combine them.	$2\sqrt[4]{3xy}$

TRY IT : : 8.73 Simplify: ⓐ $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$ ⓑ $4\sqrt[4]{5xy}+2\sqrt[4]{5xy}-7\sqrt[4]{5xy}$.

TRY IT : : 8.74 Simplify: ⓐ $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$ ⓑ $6\sqrt[3]{7mn}+\sqrt[3]{7mn}-4\sqrt[3]{7mn}$.

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

EXAMPLE 8.38

Simplify: ⓐ $\sqrt{20}+3\sqrt{5}$ ⓑ $\sqrt[3]{24}-\sqrt[3]{375}$ ⓒ $\frac{1}{2}\sqrt[4]{48}-\frac{2}{3}\sqrt[4]{243}$.

Solution

ⓐ

	$\sqrt{20}+3\sqrt{5}$
Simplify the radicals, when possible.	$\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$
	$2\sqrt{5}+3\sqrt{5}$
Combine the like radicals.	$5\sqrt{5}$

ⓑ

	$\sqrt[3]{24}-\sqrt[3]{375}$
Simplify the radicals.	$\sqrt[3]{8}\cdot\sqrt[3]{3}-\sqrt[3]{125}\cdot\sqrt[3]{3}$
	$2\sqrt[3]{3}-5\sqrt[3]{3}$
Combine the like radicals.	$-3\sqrt[3]{3}$

ⓒ

	$\frac{1}{2}\sqrt[4]{48} - \frac{2}{3}\sqrt[4]{243}$
Simplify the radicals.	$\frac{1}{2}\sqrt[4]{16}\cdot\sqrt[4]{3} - \frac{2}{3}\sqrt[4]{81}\cdot\sqrt[4]{3}$
	$\frac{1}{2}\cdot 2\cdot\sqrt[4]{3} - \frac{2}{3}\cdot 3\cdot\sqrt[4]{3}$
	$\sqrt[4]{3} - 2\sqrt[4]{3}$
Combine the like radicals.	$-\sqrt[4]{3}$

TRY IT : : 8.75 Simplify: ⓐ $\sqrt{18} + 6\sqrt{2}$ ⓑ $6\sqrt[3]{16} - 2\sqrt[3]{250}$ ⓒ $\frac{2}{3}\sqrt[3]{81} - \frac{1}{2}\sqrt[3]{24}$.

TRY IT : : 8.76 Simplify: ⓐ $\sqrt{27} + 4\sqrt{3}$ ⓑ $4\sqrt[3]{5} - 7\sqrt[3]{40}$ ⓒ $\frac{1}{2}\sqrt[3]{128} - \frac{5}{3}\sqrt[3]{54}$.

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

EXAMPLE 8.39

Simplify: ⓐ $9\sqrt{50m^2} - 6\sqrt{48m^2}$ ⓑ $\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$.

Solution

ⓐ

	$9\sqrt{50m^2} - 6\sqrt{48m^2}$
Simplify the radicals.	$9\sqrt{25m^2}\cdot\sqrt{2} - 6\sqrt{16m^2}\cdot\sqrt{3}$
	$9\cdot 5m\cdot\sqrt{2} - 6\cdot 4m\cdot\sqrt{3}$
	$45m\sqrt{2} - 24m\sqrt{3}$
The radicals are not like and so cannot be combined.	

ⓑ

	$\sqrt[3]{54n^5} - \sqrt[3]{16n^5}$
Simplify the radicals.	$\sqrt[3]{27n^3}\cdot\sqrt[3]{2n^2} - \sqrt[3]{8n^3}\cdot\sqrt[3]{2n^2}$
	$3n\sqrt[3]{2n^2} - 2n\sqrt[3]{2n^2}$
Combine the like radicals.	$n\sqrt[3]{2n^2}$

TRY IT : : 8.77 Simplify: ⓐ $\sqrt{32m^7} - \sqrt{50m^7}$ ⓑ $\sqrt[3]{135x^7} - \sqrt[3]{40x^7}$.

TRY IT : : 8.78 Simplify: ⓐ $\sqrt{27p^3} - \sqrt{48p^3}$ ⓑ $\sqrt[3]{256y^5} - \sqrt[3]{32n^5}$.

Multiply Radical Expressions

We have used the Product Property of Roots to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots 'in reverse' to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Product Property of Roots

For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

When we multiply two radicals they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is 12*xy*. Keep this in mind as you do these examples.

EXAMPLE 8.40

Simplify: ⓐ $(6\sqrt{2})(3\sqrt{10})$ ⓑ $\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$.

Solution

ⓐ

	$(6\sqrt{2})(3\sqrt{10})$
Multiply using the Product Property.	$18\sqrt{20}$
Simplify the radical.	$18\sqrt{4} \cdot \sqrt{5}$
Simplify.	$18 \cdot 2 \cdot \sqrt{5}$
	$36\sqrt{5}$

ⓑ

	$\left(-5\sqrt[3]{4}\right)\left(-4\sqrt[3]{6}\right)$
Multiply using the Product Property.	$20\sqrt[3]{24}$
Simplify the radical.	$20\sqrt[3]{8} \cdot \sqrt[3]{3}$
Simplify.	$20 \cdot 2 \cdot \sqrt[3]{3}$
	$40\sqrt[3]{3}$

TRY IT : : 8.79 Simplify: ⓐ $(3\sqrt{2})(2\sqrt{30})$ ⓑ $\left(2\sqrt[3]{18}\right)\left(-3\sqrt[3]{6}\right)$.

TRY IT : : 8.80 Simplify: ⓐ $(3\sqrt{3})(3\sqrt{6})$ ⓑ $\left(-4\sqrt[3]{9}\right)\left(3\sqrt[3]{6}\right)$.

We follow the same procedures when there are variables in the radicands.

EXAMPLE 8.41

Simplify: ⓐ $\left(10\sqrt{6p^3}\right)(4\sqrt{3p})$ ⓑ $\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$.

Solution

ⓐ

	$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
Multiply.	$40\sqrt{18p^4}$
Simplify the radical.	$40\sqrt{9p^4}\cdot\sqrt{2}$
Simplify.	$40\cdot 3p^2\cdot\sqrt{3}$
	$120p^2\sqrt{3}$

ⓑ When the radicands involve large numbers, it is often advantageous to factor them in order to find the perfect powers.

	$\left(2\sqrt[4]{20y^2}\right)\left(3\sqrt[4]{28y^3}\right)$
Multiply.	$6\sqrt[4]{4\cdot 5\cdot 4\cdot 7y^5}$
Simplify the radical.	$6\sqrt[4]{16y^4}\cdot\sqrt[4]{35y}$
Simplify.	$6\cdot 2y\sqrt[4]{35y}$
Multiply.	$12y\sqrt[4]{35y}$

TRY IT : : 8.81 Simplify: ⓐ $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$ ⓑ $\left(-4\sqrt[4]{12y^3}\right)\left(-\sqrt[4]{8y^3}\right)$.

TRY IT : : 8.82 Simplify: ⓐ $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$ ⓑ $\left(-4\sqrt[4]{9a^3}\right)\left(3\sqrt[4]{27a^2}\right)$.

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the Distributive Property to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

EXAMPLE 8.42

Simplify: ⓐ $\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$ ⓑ $\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$.

Solution

ⓐ

	$\sqrt{6}\left(\sqrt{2}+\sqrt{18}\right)$
Multiply.	$\sqrt{12}+\sqrt{108}$
Simplify.	$\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$
Simplify.	$2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$8\sqrt{3}$

ⓑ

	$\sqrt[3]{9}\left(5-\sqrt[3]{18}\right)$
Distribute.	$5\sqrt[3]{9}-\sqrt[3]{162}$
Simplify.	$5\sqrt[3]{9}-\sqrt[3]{27}\cdot\sqrt[3]{6}$
Simplify.	$5\sqrt[3]{9}-3\sqrt[3]{6}$

TRY IT : : 8.83 Simplify: ⓐ $\sqrt{6}(1+3\sqrt{6})$ ⓑ $\sqrt[3]{4}\left(-2-\sqrt[3]{6}\right)$.

TRY IT : : 8.84 Simplify: ⓐ $\sqrt{8}(2-5\sqrt{8})$ ⓑ $\sqrt[3]{3}\left(-\sqrt[3]{9}-\sqrt[3]{6}\right)$.

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

EXAMPLE 8.43

Simplify: ⓐ $(3-2\sqrt{7})(4-2\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$.

Solution

ⓐ

	$(3-2\sqrt{7})(4-2\sqrt{7})$
Multiply	$12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$
Simplify.	$12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$40-14\sqrt{7}$

ⓑ

	$\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}+4\right)$
Multiply.	$\sqrt[3]{x^2}+4\sqrt[3]{x}-2\sqrt[3]{x}-8$
Combine like terms.	$\sqrt[3]{x^2}+2\sqrt[3]{x}-8$

TRY IT : : 8.85 Simplify: ⓐ $(6-3\sqrt{7})(3+4\sqrt{7})$ ⓑ $\left(\sqrt[3]{x}-2\right)\left(\sqrt[3]{x}-3\right)$.

TRY IT : : 8.86 Simplify: ⓐ $(2-3\sqrt{11})(4-\sqrt{11})$ ⓑ $\left(\sqrt[3]{x}+1\right)\left(\sqrt[3]{x}+3\right)$.

EXAMPLE 8.44

Simplify: $\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$.

Solution

	$\left(3\sqrt{2}-\sqrt{5}\right)\left(\sqrt{2}+4\sqrt{5}\right)$
Multiply.	$3\cdot 2+12\sqrt{10}-\sqrt{10}-4\cdot 5$
Simplify.	$6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$-14+11\sqrt{10}$

TRY IT : : 8.87 Simplify: $\left(5\sqrt{3}-\sqrt{7}\right)\left(\sqrt{3}+2\sqrt{7}\right)$

TRY IT : : 8.88 Simplify: $\left(\sqrt{6}-3\sqrt{8}\right)\left(2\sqrt{6}+\sqrt{8}\right)$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)(a-b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

We will use the special product formulas in the next few examples. We will start with the Product of Binomial Squares Pattern.

EXAMPLE 8.45

Simplify: ⓐ $\left(2+\sqrt{3}\right)^2$ ⓑ $\left(4-2\sqrt{5}\right)^2$.

Solution

Be sure to include the $2ab$ term when squaring a binomial.

ⓐ

	$(a+b)^2$ $\left(2+\sqrt{3}\right)^2$
Multiply, using the Product of Binomial Squares Pattern.	$a^2 + 2\ a\ b + b^2$ $2^2 + 2 \cdot 2 \cdot \sqrt{3} + \left(\sqrt{3}\right)^2$
Simplify.	$4 + 4\sqrt{3} + 3$
Combine like terms.	$7 + 4\sqrt{3}$

ⓑ

	$(a-b)^2$ $\left(4-2\sqrt{5}\right)^2$
Multiply, using the Product of Binomial Squares Pattern.	$a^2 - 2\ a\ b + b^2$ $4^2 - 2 \cdot 4 \cdot 2\sqrt{5} + \left(2\sqrt{5}\right)^2$
Simplify.	$16 - 16\sqrt{5} + 4 \cdot 5$
	$16 - 16\sqrt{5} + 20$
Combine like terms.	$36 - 16\sqrt{5}$

TRY IT : : 8.89 Simplify: ⓐ $\left(10+\sqrt{2}\right)^2$ ⓑ $\left(1+3\sqrt{6}\right)^2$.

TRY IT : : 8.90 Simplify: ⓐ $\left(6-\sqrt{5}\right)^2$ ⓑ $\left(9-2\sqrt{10}\right)^2$.

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

EXAMPLE 8.46

Simplify: $\left(5-2\sqrt{3}\right)\left(5+2\sqrt{3}\right)$.

Solution

	$(a - b)(a + b)$ $(5 - 2\sqrt{3})(5 + 2\sqrt{3})$
Multiply, using the Product of Conjugates Pattern.	$a^2 - b^2$ $5^2 - (2\sqrt{3})^2$
Simplify.	$25 - 4 \cdot 3$
	13

TRY IT : : 8.91 Simplify: $(3 - 2\sqrt{5})(3 + 2\sqrt{5})$

TRY IT : : 8.92 Simplify: $(4 + 5\sqrt{7})(4 - 5\sqrt{7})$.

MEDIA : :

Access these online resources for additional instruction and practice with adding, subtracting, and multiplying radical expressions.

- **Multiplying Adding Subtracting Radicals (https://openstax.org/l/37Radicals1)**
- **Multiplying Special Products: Square Binomials Containing Square Roots (https://openstax.org/l/37Radicals2)**
- **Multiplying Conjugates (https://openstax.org/l/37Radicals3)**

8.4 EXERCISES

Practice Makes Perfect

Add and Subtract Radical Expressions

In the following exercises, simplify.

165.
ⓐ $8\sqrt{2}-5\sqrt{2}$
ⓑ $5\sqrt[3]{m}+2\sqrt[3]{m}$
ⓒ $8\sqrt[4]{m}-2\sqrt[4]{n}$

166.
ⓐ $7\sqrt{2}-3\sqrt{2}$
ⓑ $7\sqrt[3]{p}+2\sqrt[3]{p}$
ⓒ $5\sqrt[3]{x}-3\sqrt[3]{x}$

167.
ⓐ $3\sqrt{5}+6\sqrt{5}$
ⓑ $9\sqrt[3]{a}+3\sqrt[3]{a}$
ⓒ $5\sqrt[4]{2z}+\sqrt[4]{2z}$

168.
ⓐ $4\sqrt{5}+8\sqrt{5}$
ⓑ $\sqrt[3]{m}-4\sqrt[3]{m}$
ⓒ $\sqrt{n}+3\sqrt{n}$

169.
ⓐ $3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$
ⓑ $5\sqrt[4]{3ab}-3\sqrt[4]{3ab}-2\sqrt[4]{3ab}$

170.
ⓐ $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$
ⓑ $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$

171.
ⓐ $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$
ⓑ $2\sqrt[3]{4pq}-5\sqrt[3]{4pq}+4\sqrt[3]{4pq}$

172.
ⓐ $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$
ⓑ $11\sqrt[3]{2rs}-9\sqrt[3]{2rs}+3\sqrt[3]{2rs}$

173.
ⓐ $\sqrt{27}-\sqrt{75}$
ⓑ $\sqrt[3]{40}-\sqrt[3]{320}$
ⓒ $\frac{1}{2}\sqrt[4]{32}+\frac{2}{3}\sqrt[4]{162}$

174.
ⓐ $\sqrt{72}-\sqrt{98}$
ⓑ $\sqrt[3]{24}+\sqrt[3]{81}$
ⓒ $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$

175.
ⓐ $\sqrt{48}+\sqrt{27}$
ⓑ $\sqrt[3]{54}+\sqrt[3]{128}$
ⓒ $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$

176.
ⓐ $\sqrt{45}+\sqrt{80}$
ⓑ $\sqrt[3]{81}-\sqrt[3]{192}$
ⓒ $\frac{5}{2}\sqrt[4]{80}+\frac{7}{3}\sqrt[4]{405}$

177.
ⓐ $\sqrt{72a^5}-\sqrt{50a^5}$
ⓑ $9\sqrt[4]{80p^4}-6\sqrt[4]{405p^4}$

178.
ⓐ $\sqrt{48b^5}-\sqrt{75b^5}$
ⓑ $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$

179.
ⓐ $\sqrt{80c^7}-\sqrt{20c^7}$
ⓑ $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$

180.
ⓐ $\sqrt{96d^9}-\sqrt{24d^9}$
ⓑ $5\sqrt[4]{243s^6}+2\sqrt[4]{3s^6}$

181.
$3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$

182. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Multiply Radical Expressions

In the following exercises, simplify.

183.
ⓐ $(-2\sqrt{3})(3\sqrt{18})$
ⓑ $\left(8\sqrt[3]{4}\right)\left(-4\sqrt[3]{18}\right)$

184.
ⓐ $(-4\sqrt{5})(5\sqrt{10})$
ⓑ $\left(-2\sqrt[3]{9}\right)\left(7\sqrt[3]{9}\right)$

185.
ⓐ $(5\sqrt{6})(-\sqrt{12})$
ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

186.
ⓐ $(-2\sqrt{7})(-2\sqrt{14})$
ⓑ $\left(-3\sqrt[4]{8}\right)\left(-5\sqrt[4]{6}\right)$

187.
ⓐ $\left(4\sqrt{12z^3}\right)(3\sqrt{9z})$
ⓑ $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

188.
ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

189.
ⓐ $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$
ⓑ $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

190.
ⓐ $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$
ⓑ $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

191.
ⓐ $\sqrt{7}(5+2\sqrt{7})$
ⓑ $\sqrt[3]{6}\left(4+\sqrt[3]{18}\right)$

192.
ⓐ $\sqrt{11}(8+4\sqrt{11})$
ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9}+\sqrt[3]{18}\right)$

193.
ⓐ $\sqrt{11}(-3+4\sqrt{11})$
ⓑ $\sqrt[4]{3}\left(\sqrt[4]{54}+\sqrt[4]{18}\right)$

194.
ⓐ $\sqrt{2}(-5+9\sqrt{2})$
ⓑ $\sqrt[4]{2}\left(\sqrt[4]{12}+\sqrt[4]{24}\right)$

195. $(7+\sqrt{3})(9-\sqrt{3})$

196. $(8-\sqrt{2})(3+\sqrt{2})$

197.
ⓐ $(9-3\sqrt{2})(6+4\sqrt{2})$
ⓑ $\left(\sqrt[3]{x}-3\right)\left(\sqrt[3]{x}+1\right)$

198.
ⓐ $(3-2\sqrt{7})(5-4\sqrt{7})$
ⓑ $\left(\sqrt[3]{x}-5\right)\left(\sqrt[3]{x}-3\right)$

199.
ⓐ $(1+3\sqrt{10})(5-2\sqrt{10})$
ⓑ $\left(2\sqrt[3]{x}+6\right)\left(\sqrt[3]{x}+1\right)$

200.
ⓐ $(7-2\sqrt{5})(4+9\sqrt{5})$
ⓑ $\left(3\sqrt[3]{x}+2\right)\left(\sqrt[3]{x}-2\right)$

201. $(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$

202. $(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

203. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$

204. $(4\sqrt{6}+7\sqrt{13})(8\sqrt{6}-3\sqrt{13})$

205. ⓐ $(3+\sqrt{5})^2$ ⓑ $(2-5\sqrt{3})^2$

206. ⓐ $(4+\sqrt{11})^2$ ⓑ $(3-2\sqrt{5})^2$

207. ⓐ $(9-\sqrt{6})^2$ ⓑ $(10+3\sqrt{7})^2$

208. ⓐ $(5-\sqrt{10})^2$ ⓑ $(8+3\sqrt{2})^2$

209. $(4+\sqrt{2})(4-\sqrt{2})$

210. $(7+\sqrt{10})(7-\sqrt{10})$

211. $(4+9\sqrt{3})(4-9\sqrt{3})$

212. $(1+8\sqrt{2})(1-8\sqrt{2})$

213. $(12-5\sqrt{5})(12+5\sqrt{5})$

214. $(9-4\sqrt{3})(9+4\sqrt{3})$

215. $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

216. $\left(\sqrt[3]{4x}+3\right)\left(\sqrt[3]{4x}-3\right)$

Mixed Practice

217. $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$

218. $\sqrt{175k^4}-\sqrt{63k^4}$

219. $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$

220. $\sqrt[3]{24} + \sqrt[3]{/81}$

221. $\frac{1}{2}\sqrt[4]{80} - \frac{2}{3}\sqrt[4]{405}$

222. $8\sqrt[4]{13} - 4\sqrt[4]{13} - 3\sqrt[4]{13}$

223. $5\sqrt{12c^4} - 3\sqrt{27c^6}$

224. $\sqrt{80a^5} - \sqrt{45a^5}$

225. $\frac{3}{5}\sqrt{75} - \frac{1}{4}\sqrt{48}$

226. $21\sqrt[3]{9} - 2\sqrt[3]{9}$

227. $8\sqrt[3]{64q^6} - 3\sqrt[3]{125q^6}$

228. $11\sqrt{11} - 10\sqrt{11}$

229. $\sqrt{3} \cdot \sqrt{21}$

230. $(4\sqrt{6})(-\sqrt{18})$

231. $\left(7\sqrt[3]{4}\right)\left(-3\sqrt[3]{18}\right)$

232. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

233. $(\sqrt{29})^2$

234. $(-4\sqrt{17})(-3\sqrt{17})$

235. $(-4 + \sqrt{17})(-3 + \sqrt{17})$

236. $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

237. $(6 - 3\sqrt{2})^2$

238. $\sqrt{3}(4 - 3\sqrt{3})$

239. $\sqrt[3]{3}\left(2\sqrt[3]{9} + \sqrt[3]{18}\right)$

240. $(\sqrt{6} + \sqrt{3})(\sqrt{6} + 6\sqrt{3})$

Writing Exercises

241. Explain the when a radical expression is in simplest form.

242. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.

243.

ⓐ Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.

ⓑ Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.

244. Use the binomial square pattern to simplify $(3 + \sqrt{2})^2$. Explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

8.5 Divide Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\frac{30}{48}$.
   If you missed this problem, review Example 1.24.
2. Simplify: $x^2 \cdot x^4$.
   If you missed this problem, review Example 5.12.
3. Multiply: $(7+3x)(7-3x)$.
   If you missed this problem, review Example 5.32.

Divide Radical Expressions

We have used the Quotient Property of Radical Expressions to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals.

We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Quotient Property of Radical Expressions

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{and} \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

EXAMPLE 8.47

Simplify: ⓐ $\frac{\sqrt{72x^3}}{\sqrt{162x}}$ ⓑ $\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$.

Solution

ⓐ

	$\frac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.	$\sqrt{\frac{72x^3}{162x}}$
Remove common factors.	$\sqrt{\frac{\cancel{18} \cdot 4 \cdot x^2 \cdot \cancel{x}}{\cancel{18} \cdot 9 \cdot \cancel{x}}}$
Simplify.	$\sqrt{\frac{4x^2}{9}}$
Simplify the radical.	$\frac{2x}{3}$

ⓑ

	$\frac{\sqrt[3]{32x^2}}{\sqrt[3]{4x^5}}$
Rewrite using the quotient property, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$.	$\sqrt[3]{\frac{32x^2}{4x^5}}$
Simplify the fraction under the radical.	$\sqrt[3]{\frac{8}{x^3}}$
Simplify the radical.	$\frac{2}{x}$

TRY IT : : 8.93

Simplify: ⓐ $\frac{\sqrt{50s^3}}{\sqrt{128s}}$ ⓑ $\frac{\sqrt[3]{56a}}{\sqrt[3]{7a^4}}$.

TRY IT : : 8.94

Simplify: ⓐ $\frac{\sqrt{75q^5}}{\sqrt{108q}}$ ⓑ $\frac{\sqrt[3]{72b^2}}{\sqrt[3]{9b^5}}$.

EXAMPLE 8.48

Simplify: ⓐ $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$ ⓑ $\frac{\sqrt[3]{-250mn^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$.

Solution

ⓐ

	$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property.	$\sqrt{\frac{147ab^8}{3a^3b^4}}$
Remove common factors in the fraction.	$\sqrt{\frac{49b^4}{a^2}}$
Simplify the radical.	$\frac{7b^2}{a}$

ⓑ

	$\dfrac{\sqrt[3]{-250m\,n^{-2}}}{\sqrt[3]{2m^{-2}n^4}}$
Rewrite using the quotient property.	$\sqrt[3]{\dfrac{-250m\,n^{-2}}{2m^{-2}n^4}}$
Simplify the fraction under the radical.	$\sqrt[3]{\dfrac{-125m^3}{n^6}}$
Simplify the radical.	$-\dfrac{5m}{n^2}$

> **TRY IT : : 8.95**
>
> Simplify: ⓐ $\dfrac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$ ⓑ $\dfrac{\sqrt[3]{-128x^2y^{-1}}}{\sqrt[3]{2x^{-1}y^2}}$.

> **TRY IT : : 8.96**
>
> Simplify: ⓐ $\dfrac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$ ⓑ $\dfrac{\sqrt[3]{-81pq^{-1}}}{\sqrt[3]{3p^{-2}q^5}}$.

EXAMPLE 8.49

Simplify: $\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$.

Solution

	$\dfrac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$
Rewrite using the quotient property.	$\sqrt{\dfrac{54x^5y^3}{3x^2y}}$
Remove common factors in the fraction.	$\sqrt{18x^3y^2}$
Rewrite the radicand as a product using the largest perfect square factor.	$\sqrt{9x^2y^2 \cdot 2x}$
Rewrite the radical as the product of two radicals.	$\sqrt{9x^2y^2} \cdot \sqrt{2x}$
Simplify.	$3xy\sqrt{2x}$

> **TRY IT : : 8.97**
>
> Simplify: $\dfrac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

> **TRY IT : : 8.98**
>
> Simplify: $\dfrac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$.

Rationalize a One Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process!

For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator.

This process is still used today, and is useful in other areas of mathematics, too.

Rationalizing the Denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator still must be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a radical expression is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

A radical expression is considered simplified if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

EXAMPLE 8.50

Simplify: ⓐ $\frac{4}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{20}}$ ⓒ $\frac{3}{\sqrt{6x}}$.

Solution

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$\frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$\frac{4\sqrt{3}}{3}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt{\frac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$\frac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$\frac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$\frac{\sqrt{3} \cdot \sqrt{5}}{2\sqrt{5} \cdot \sqrt{5}}$

Simplify.	$\frac{\sqrt{15}}{2 \cdot 5}$
Simplify.	$\frac{\sqrt{15}}{10}$

ⓒ

	$\frac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$\frac{3 \cdot \sqrt{6x}}{\sqrt{6x} \cdot \sqrt{6x}}$
Simplify.	$\frac{3\sqrt{6x}}{6x}$
Simplify.	$\frac{\sqrt{6x}}{2x}$

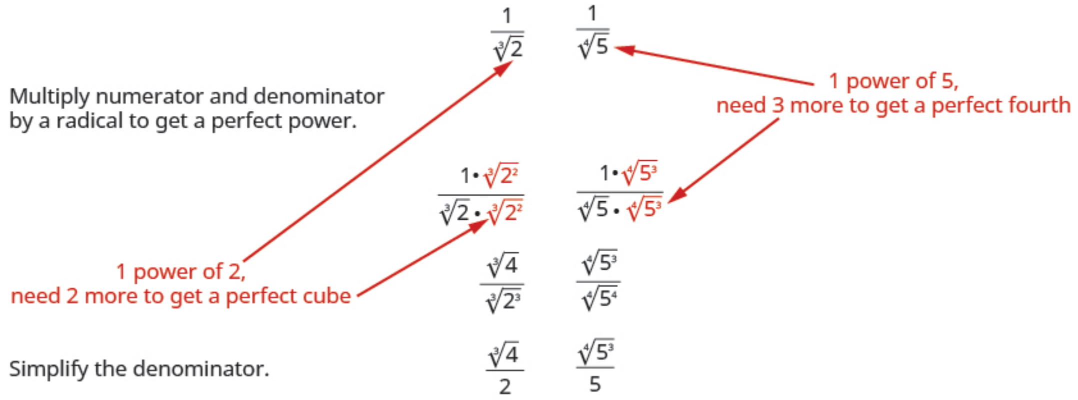
TRY IT : : 8.99 Simplify: ⓐ $\frac{5}{\sqrt{3}}$ ⓑ $\sqrt{\frac{3}{32}}$ ⓒ $\frac{2}{\sqrt{2x}}$.

TRY IT : : 8.100 Simplify: ⓐ $\frac{6}{\sqrt{5}}$ ⓑ $\sqrt{\frac{7}{18}}$ ⓒ $\frac{5}{\sqrt{5x}}$.

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

We will follow a similar process to rationalize higher roots. To rationalize a denominator with a higher index radical, we multiply the numerator and denominator by a radical that would give us a radicand that is a perfect power of the index. When we simplify the new radical, the denominator will no longer have a radical.

For example,



We will use this technique in the next examples.

EXAMPLE 8.51

Simplify ⓐ $\frac{1}{\sqrt[3]{6}}$ ⓑ $\sqrt[3]{\frac{7}{24}}$ ⓒ $\frac{3}{\sqrt[3]{4x}}$.

Solution

To rationalize a denominator with a cube root, we can multiply by a cube root that will give us a perfect cube in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{1}{\sqrt[3]{6}}$
The radical in the denominator has one factor of 6. Multiply both the numerator and denominator by $\sqrt[3]{6^2}$, which gives us 2 more factors of 6.	$\frac{1 \cdot \sqrt[3]{6^2}}{\sqrt[3]{6} \cdot \sqrt[3]{6^2}}$
Multiply. Notice the radicand in the denominator has 3 powers of 6.	$\frac{\sqrt[3]{6^2}}{\sqrt[3]{6^3}}$
Simplify the cube root in the denominator.	$\frac{\sqrt[3]{36}}{6}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt[3]{\frac{7}{24}}$
The fraction is not a perfect cube, so rewrite using the Quotient Property.	$\frac{\sqrt[3]{7}}{\sqrt[3]{24}}$
Simplify the denominator.	$\frac{\sqrt[3]{7}}{2\sqrt[3]{3}}$
Multiply the numerator and denominator by $\sqrt[3]{3^2}$. This will give us 3 factors of 3.	$\frac{\sqrt[3]{7} \cdot \sqrt[3]{3^2}}{2\sqrt[3]{3} \cdot \sqrt[3]{3^2}}$
Simplify.	$\frac{\sqrt[3]{63}}{2\sqrt[3]{3^3}}$
Remember, $\sqrt[3]{3^3} = 3$.	$\frac{\sqrt[3]{63}}{2 \cdot 3}$
Simplify.	$\frac{\sqrt[3]{63}}{6}$

ⓒ

	$\frac{3}{\sqrt[3]{4x}}$
Rewrite the radicand to show the factors.	$\frac{3}{\sqrt[3]{2^2 \cdot x}}$
Multiply the numerator and denominator by $\sqrt[3]{2 \cdot x^2}$. This will get us 3 factors of 2 and 3 factors of x.	$\frac{3 \cdot \sqrt[3]{2 \cdot x^2}}{\sqrt[3]{2^2 x} \cdot \sqrt[3]{2 \cdot x^2}}$
Simplify.	$\frac{3\sqrt[3]{2x^2}}{\sqrt[3]{2^3 x^3}}$
Simplify the radical in the denominator.	$\frac{3\sqrt[3]{2x^2}}{2x}$

TRY IT : : 8.101 Simplify: ⓐ $\frac{1}{\sqrt[3]{7}}$ ⓑ $\sqrt[3]{\frac{5}{12}}$ ⓒ $\frac{5}{\sqrt[3]{9y}}$.

TRY IT : : 8.102 Simplify: ⓐ $\frac{1}{\sqrt[3]{2}}$ ⓑ $\sqrt[3]{\frac{3}{20}}$ ⓒ $\frac{2}{\sqrt[3]{25n}}$.

EXAMPLE 8.52

Simplify: ⓐ $\frac{1}{\sqrt[4]{2}}$ ⓑ $\sqrt[4]{\frac{5}{64}}$ ⓒ $\frac{2}{\sqrt[4]{8x}}$.

Solution

To rationalize a denominator with a fourth root, we can multiply by a fourth root that will give us a perfect fourth power in the radicand in the denominator. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

ⓐ

	$\frac{1}{\sqrt[4]{2}}$
The radical in the denominator has one factor of 2. Multiply both the numerator and denominator by $\sqrt[4]{2^3}$, which gives us 3 more factors of 2.	$\frac{1 \cdot \sqrt[4]{2^3}}{\sqrt[4]{2} \cdot \sqrt[4]{2^3}}$
Multiply. Notice the radicand in the denominator has 4 powers of 2.	$\frac{\sqrt[4]{8}}{\sqrt[4]{2^4}}$
Simplify the fourth root in the denominator.	$\frac{\sqrt[4]{8}}{2}$

ⓑ We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt[4]{\frac{5}{64}}$

The fraction is not a perfect fourth power, so rewrite using the Quotient Property.	$\frac{\sqrt[4]{5}}{\sqrt[4]{64}}$
Rewrite the radicand in the denominator to show the factors.	$\frac{\sqrt[4]{5}}{\sqrt[4]{2^6}}$
Simplify the denominator.	$\frac{\sqrt[4]{5}}{2\sqrt[4]{2^2}}$
Multiply the numerator and denominator by $\sqrt[4]{2^2}$. This will give us 4 factors of 2.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{2^2}}{2\sqrt[4]{2^2}\cdot\sqrt[4]{2^2}}$
Simplify.	$\frac{\sqrt[4]{5}\cdot\sqrt[4]{4}}{2\sqrt[4]{2^4}}$
Remember, $\sqrt[4]{2^4} = 2$.	$\frac{\sqrt[4]{20}}{2\cdot 2}$
Simplify.	$\frac{\sqrt[4]{20}}{4}$

ⓒ

	$\frac{2}{\sqrt[4]{8x}}$
Rewrite the radicand to show the factors.	$\frac{2}{\sqrt[4]{2^3\cdot x}}$
Multiply the numerator and denominator by $\sqrt[4]{2\cdot x^3}$. This will get us 4 factors of 2 and 4 factors of x.	$\frac{2\cdot\sqrt[4]{2\cdot x^3}}{\sqrt[4]{2^3x}\cdot\sqrt[4]{2\cdot x^3}}$
Simplify.	$\frac{2\sqrt[4]{2x^3}}{\sqrt[4]{2^4x^4}}$
Simplify the radical in the denominator.	$\frac{2\sqrt[4]{2x^3}}{2^4x^4}$
Simplify the fraction.	$\frac{\sqrt[4]{2x^3}}{x}$

> **TRY IT : :** 8.103 Simplify: ⓐ $\frac{1}{\sqrt[4]{3}}$ ⓑ $\sqrt[4]{\frac{3}{64}}$ ⓒ $\frac{3}{\sqrt[4]{125x}}$.

> **TRY IT : :** 8.104 Simplify: ⓐ $\frac{1}{\sqrt[4]{5}}$ ⓑ $\sqrt[4]{\frac{7}{128}}$ ⓒ $\frac{4}{\sqrt[4]{4x}}$

Rationalize a Two Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the Product of Conjugates Pattern to rationalize the denominator.

$$(a-b)(a+b) \qquad (2-\sqrt{5})(2+\sqrt{5})$$
$$a^2-b^2 \qquad 2^2-(\sqrt{5})^2$$
$$4-5$$
$$-1$$

When we multiply a binomial that includes a square root by its conjugate, the product has no square roots.

EXAMPLE 8.53

Simplify: $\frac{5}{2-\sqrt{3}}$.

Solution

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{5(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$\frac{5(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$\frac{5(2+\sqrt{3})}{1}$
Simplify.	$5(2+\sqrt{3})$

TRY IT : : 8.105 Simplify: $\frac{3}{1-\sqrt{5}}$.

TRY IT : : 8.106 Simplify: $\frac{2}{4-\sqrt{6}}$.

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

EXAMPLE 8.54

Simplify: $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$

Simplify the denominator.	$\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

TRY IT : : 8.107 Simplify: $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

TRY IT : : 8.108 Simplify: $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

EXAMPLE 8.55

Simplify: $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

TRY IT : : 8.109 Simplify: $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

TRY IT : : 8.110 Simplify: $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$

MEDIA : :

Access these online resources for additional instruction and practice with dividing radical expressions.

- Rationalize the Denominator (https://openstax.org/l/37RatDenom1)
- Dividing Radical Expressions and Rationalizing the Denominator (https://openstax.org/l/37RatDenom2)
- Simplifying a Radical Expression with a Conjugate (https://openstax.org/l/37RatDenom3)
- Rationalize the Denominator of a Radical Expression (https://openstax.org/l/37RatDenom4)

8.5 EXERCISES

Practice Makes Perfect

Divide Square Roots

In the following exercises, simplify.

245. ⓐ $\frac{\sqrt{128}}{\sqrt{72}}$ ⓑ $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$

246. ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$ ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

247. ⓐ $\frac{\sqrt{200m^5}}{\sqrt{98m}}$ ⓑ $\frac{\sqrt[3]{54y^2}}{\sqrt[3]{2y^5}}$

248. ⓐ $\frac{\sqrt{108n^7}}{\sqrt{243n^3}}$ ⓑ $\frac{\sqrt[3]{54y}}{\sqrt[3]{16y^4}}$

249. ⓐ $\frac{\sqrt{75r^3}}{\sqrt{108r^7}}$ ⓑ $\frac{\sqrt[3]{24x^7}}{\sqrt[3]{81x^4}}$

250. ⓐ $\frac{\sqrt{196q}}{\sqrt{484q^5}}$ ⓑ $\frac{\sqrt[3]{16m^4}}{\sqrt[3]{54m}}$

251. ⓐ $\frac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$ ⓑ $\frac{\sqrt[3]{-16a^4b^{-2}}}{\sqrt[3]{2a^{-2}b}}$

252. ⓐ $\frac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$ ⓑ $\frac{\sqrt[3]{-375y^4z^{-2}}}{\sqrt[3]{3y^{-2}z^4}}$

253. ⓐ $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$ ⓑ $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

254. ⓐ $\frac{\sqrt{810c^{-3}d^7}}{\sqrt{1000c\,d^{-1}}}$ ⓑ $\frac{\sqrt[3]{24a^7b^{-1}}}{\sqrt[3]{-81a^{-2}b^2}}$

255. $\frac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$

256. $\frac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$

257. $\frac{\sqrt[3]{48a^3b^6}}{\sqrt[3]{3a^{-1}b^3}}$

258. $\frac{\sqrt[3]{162x^{-3}y^6}}{\sqrt[3]{2x^3y^{-2}}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

259. ⓐ $\frac{10}{\sqrt{6}}$ ⓑ $\sqrt{\frac{4}{27}}$ ⓒ $\frac{10}{\sqrt{5x}}$

260. ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

261. ⓐ $\frac{6}{\sqrt{7}}$ ⓑ $\sqrt{\frac{8}{45}}$ ⓒ $\frac{12}{\sqrt{3p}}$

262. ⓐ $\frac{4}{\sqrt{5}}$ ⓑ $\sqrt{\frac{27}{80}}$ ⓒ $\frac{18}{\sqrt{6q}}$

263. ⓐ $\frac{1}{\sqrt[3]{5}}$ ⓑ $\sqrt[3]{\frac{5}{24}}$ ⓒ $\frac{4}{\sqrt[3]{36a}}$

264. ⓐ $\frac{1}{\sqrt[3]{3}}$ ⓑ $\sqrt[3]{\frac{5}{32}}$ ⓒ $\frac{7}{\sqrt[3]{49b}}$

265. ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^2}}$

266. ⓐ $\frac{1}{\sqrt[3]{13}}$ ⓑ $\sqrt[3]{\frac{3}{128}}$ ⓒ $\frac{3}{\sqrt[3]{6y^2}}$

267. ⓐ $\frac{1}{\sqrt[4]{7}}$ ⓑ $\sqrt[4]{\frac{5}{32}}$ ⓒ $\frac{4}{\sqrt[4]{4x^2}}$

268. ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^3}}$

269. ⓐ $\frac{1}{\sqrt[4]{9}}$ ⓑ $\sqrt[4]{\frac{25}{128}}$ ⓒ $\frac{6}{\sqrt[4]{27a}}$

270. ⓐ $\frac{1}{\sqrt[4]{8}}$ ⓑ $\sqrt[4]{\frac{27}{128}}$ ⓒ $\frac{16}{\sqrt[4]{64b^2}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

271. $\frac{8}{1-\sqrt{5}}$

272. $\frac{7}{2-\sqrt{6}}$

273. $\frac{6}{3-\sqrt{7}}$

274. $\frac{5}{4-\sqrt{11}}$

275. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

276. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

277. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

278. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

279. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

280. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

281. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

282. $\frac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Writing Exercises

283.

ⓐ Simplify $\sqrt{\frac{27}{3}}$ and explain all your steps.

ⓑ Simplify $\sqrt{\frac{27}{5}}$ and explain all your steps.

ⓒ Why are the two methods of simplifying square roots different?

284. Explain what is meant by the word rationalize in the phrase, “rationalize a denominator.”

285. Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an expression with no radicals.

286. Explain why multiplying $\frac{7}{\sqrt[3]{x}}$ by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$ does not rationalize the denominator.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

ⓑ *After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?*

8.6 Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $(y-3)^2$.
   If you missed this problem, review Example 5.31.
2. Solve: $2x-5=0$.
   If you missed this problem, review Example 2.2.
3. Solve $n^2-6n+8=0$.
   If you missed this problem, review Example 6.45.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Radical Equation

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Solving radical equations containing an even index by raising both sides to the power of the index may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on raising a radical with index n to the n^{th} power. This will eliminate the radical.

$$\text{For } a \geq 0,\ \left(\sqrt[n]{a}\right)^n = a.$$

EXAMPLE 8.56 HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{5n-4}-9=0$.

Solution

Step 1. Isolate the radical on one side of the equation.	To isolate the radical, add 9 to both sides. Simplify.	$\sqrt{5n-4}-9=0$ $\sqrt{5n-4}-9+9=0+9$ $\sqrt{5n-4}=9$
Step 2. Raise both sides of the equation to the power of the index.	Since the index of a square root is 2, we square both sides.	$\left(\sqrt{5n-4}\right)^2=(9)^2$
Step 3. Solve the new equation.	Remember, $\left(\sqrt{a}\right)^2=a$.	$5n-4=81$ $5n=85$ $n=17$

Step 4. Check the answer in the original equation.		Check the answer. $\sqrt{5n-4}-9=0$ $\sqrt{5(17)-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9\stackrel{?}{=}0$ $0=0\checkmark$ The solution is $n=17$.

TRY IT : : 8.111 Solve: $\sqrt{3m+2}-5=0.$

TRY IT : : 8.112 Solve: $\sqrt{10z+1}-2=0.$

HOW TO : : SOLVE A RADICAL EQUATION WITH ONE RADICAL.

Step 1. Isolate the radical on one side of the equation.
Step 2. Raise both sides of the equation to the power of the index.
Step 3. Solve the new equation.
Step 4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a radical with an even index equal to a negative number, that equation will have no solution.

EXAMPLE 8.57

Solve: $\sqrt{9k-2}+1=0.$

Solution

	$\sqrt{9k-2}+1=0$
To isolate the radical, subtract 1 to both sides.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

TRY IT : : 8.113 Solve: $\sqrt{2r-3}+5=0.$

TRY IT : : 8.114 Solve: $\sqrt{7s-3}+2=0.$

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Binomial Squares

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

Don't forget the middle term!

EXAMPLE 8.58

Solve: $\sqrt{p-1} + 1 = p$.

Solution

	$\sqrt{p-1} + 1 = p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1} + 1 - 1 = p - 1$
Simplify.	$\sqrt{p-1} = p - 1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2 = (p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right. Then solve the new equation.	$p - 1 = p^2 - 2p + 1$
It is a quadratic equation, so get zero on one side.	$0 = p^2 - 3p + 2$
Factor the right side.	$0 = (p-1)(p-2)$
Use the Zero Product Property.	$0 = p - 1 \quad 0 = p - 2$
Solve each equation.	$p = 1 \quad p = 2$
Check the answers.	

$p = 1$: $\sqrt{p-1} + 1 = p$; $\sqrt{1-1} + 1 \stackrel{?}{=} 1$; $\sqrt{0} + 1 \stackrel{?}{=} 1$; $1 = 1\checkmark$

$p = 2$: $\sqrt{p-1} + 1 = p$; $\sqrt{2-1} + 1 \stackrel{?}{=} 2$; $\sqrt{1} + 1 \stackrel{?}{=} 2$; $2 = 2\checkmark$

The solutions are $p = 1, \quad p = 2$.

TRY IT : : 8.115 Solve: $\sqrt{x-2} + 2 = x$.

TRY IT : : 8.116 Solve: $\sqrt{y-5} + 5 = y$.

When the index of the radical is 3, we cube both sides to remove the radical.

$$\left(\sqrt[3]{a}\right)^3 = a$$

EXAMPLE 8.59

Solve: $\sqrt[3]{5x+1} + 8 = 4$.

Solution

	$\sqrt[3]{5x+1}+8=4$
To isolate the radical, subtract 8 from both sides.	$\sqrt[3]{5x+1}=-4$
Cube both sides of the equation.	$\left(\sqrt[3]{5x+1}\right)^3=(-4)^3$
Simplify.	$5x+1=-64$
Solve the equation.	$5x=-65$
	$x=-13$
Check the answer.	

$x=-13$ $\quad \sqrt[3]{5x+1}+8=4$

$\sqrt[3]{5(-13)+1}+8\stackrel{?}{=}4$

$\sqrt[3]{-64}+8\stackrel{?}{=}4$

$-4+8\stackrel{?}{=}4$

$4=4\checkmark$

The solution is $x=-13$.

TRY IT : : 8.117 Solve: $\sqrt[3]{4x-3}+8=5$

TRY IT : : 8.118 Solve: $\sqrt[3]{6x-10}+1=-3$

Sometimes an equation will contain rational exponents instead of a radical. We use the same techniques to solve the equation as when we have a radical. We raise each side of the equation to the power of the denominator of the rational exponent. Since $(a^m)^n=a^{m\cdot n}$, we have for example,

$$\left(x^{\frac{1}{2}}\right)^2=x,\quad \left(x^{\frac{1}{3}}\right)^3=x$$

Remember, $x^{\frac{1}{2}}=\sqrt{x}$ and $x^{\frac{1}{3}}=\sqrt[3]{x}$.

EXAMPLE 8.60

Solve: $(3x-2)^{\frac{1}{4}}+3=5$.

Solution

$$(3x-2)^{\frac{1}{4}}+3=5$$

To isolate the term with the rational exponent, subtract 3 from both sides.	$(3x-2)^{\frac{1}{4}}=2$
Raise each side of the equation to the fourth power.	$\left((3x-2)^{\frac{1}{4}}\right)^4=(2)^4$
Simplify.	$3x-2=16$
Solve the equation.	$3x=18$
	$x=6$
Check the answer.	
$x=6 \quad (3x-2)^{\frac{1}{4}}+3=5$ $(3\cdot 6-2)^{\frac{1}{4}}+3\stackrel{?}{=}5$ $(16)^{\frac{1}{4}}+3\stackrel{?}{=}5$ $2+3\stackrel{?}{=}5$ $5=5\checkmark$	
	The solution is $x=6$.

TRY IT : : 8.119 Solve: $(9x+9)^{\frac{1}{4}}-2=1$.

TRY IT : : 8.120 Solve: $(4x-8)^{\frac{1}{4}}+5=7$.

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an extraneous solution!

EXAMPLE 8.61

Solve: $\sqrt{r+4}-r+2=0$.

Solution

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$\left(\sqrt{r+4}\right)^2=(r-2)^2$
Simplify and then solve the equation	$r+4=r^2-4r+4$
It is a quadratic equation, so get zero on one side.	$0=r^2-5r$
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r \quad 0=r-5$

Solve the equation.	$r = 0 \quad r = 5$
Check your answer.	
$r=0,\quad \sqrt{r+4}-r+2=0$ $\sqrt{0+4}-0+2\stackrel{?}{=}0$ $\sqrt{4}+2\stackrel{?}{=}0$ $4\neq 0$ $r=5,\quad \sqrt{r+4}-r+2=0$ $\sqrt{5+4}-5+2\stackrel{?}{=}0$ $\sqrt{9}-3\stackrel{?}{=}0$ $0=0$ ✓	The solution is $r = 5$.
	$r = 0$ is an extraneous solution.

TRY IT : : 8.121 Solve: $\sqrt{m+9} - m + 3 = 0.$

TRY IT : : 8.122 Solve: $\sqrt{n+1} - n + 1 = 0.$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

EXAMPLE 8.62

Solve: $3\sqrt{3x-5} - 8 = 4.$

Solution

	$3\sqrt{3x-5} - 8 = 4$
Isolate the radical term.	$3\sqrt{3x-5} = 12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5} = 4$
Square both sides of the equation.	$\left(\sqrt{3x-5}\right)^2 = (4)^2$
Simplify, then solve the new equation.	$3x - 5 = 16$
	$3x = 21$
Solve the equation.	$x = 7$
Check the answer.	
$x=7\quad 3\sqrt{3x-5}-8=4$ $3\sqrt{3(7)-5}-8\stackrel{?}{=}4$ $3\sqrt{21-5}-8\stackrel{?}{=}4$ $3\sqrt{16}-8\stackrel{?}{=}4$ $3(4)-8\stackrel{?}{=}4$ $4=4$ ✓	
	The solution is $x = 7.$

TRY IT : : 8.123 Solve: $2\sqrt{4a+4} - 16 = 16.$

TRY IT : : 8.124 Solve: $3\sqrt{2b+3}-25=50$.

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

EXAMPLE 8.63

Solve: $\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$.

Solution

The radical terms are isolated.	$\sqrt[3]{4x-3}=\sqrt[3]{3x+2}$
Since the index is 3, cube both sides of the equation.	$\left(\sqrt[3]{4x-3}\right)^3=\left(\sqrt[3]{3x+2}\right)^3$
Simplify, then solve the new equation.	$4x-3=3x+2$
	$x-3=2$
	$x=5$
	The solution is $x=5$.

Check the answer.
We leave it to you to show that 5 checks!

TRY IT : : 8.125 Solve: $\sqrt[3]{5x-4}=\sqrt[3]{2x+5}$.

TRY IT : : 8.126 Solve: $\sqrt[3]{7x+1}=\sqrt[3]{2x-5}$.

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and raise both sides of the equation to the power of the index again.

EXAMPLE 8.64 HOW TO SOLVE A RADICAL EQUATION

Solve: $\sqrt{m}+1=\sqrt{m+9}$.

Solution

Step 1. Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Step 2. Raise both sides of the equation to the power of the index.	We square both sides. Simplify—be very careful as you multiply!	$\left(\sqrt{m}+1\right)^2=\left(\sqrt{m+9}\right)^2$
Step 3. Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$m+2\sqrt{m}+1=m+9$ $2\sqrt{m}=8$ $\sqrt{m}=4$ $\left(\sqrt{m}\right)^2=(4)^2$ $m=16$

Step 4. Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$ $\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$ $4+1\stackrel{?}{=}5$ $5=5\checkmark$ The solution is $m=16$.

TRY IT : : 8.127 Solve: $3-\sqrt{x}=\sqrt{x-3}$.

TRY IT : : 8.128 Solve: $\sqrt{x}+2=\sqrt{x+16}$.

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

HOW TO : : SOLVE A RADICAL EQUATION.

Step 1. Isolate one of the radical terms on one side of the equation.
Step 2. Raise both sides of the equation to the power of the index.
Step 3. Are there any more radicals?
If yes, repeat Step 1 and Step 2 again.
If no, solve the new equation.
Step 4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern is $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

EXAMPLE 8.65

Solve: $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution

	$\sqrt{q-2}+3=\sqrt{4q+1}$
The radical on the right is isolated. Square both sides.	$\left(\sqrt{q-2}+3\right)^2=\left(\sqrt{4q+1}\right)^2$
Simplify.	$q-2+6\sqrt{q-2}+9=4q+1$
There is still a radical in the equation so we must repeat the previous steps. Isolate the radical.	$6\sqrt{q-2}=3q-6$
Square both sides. It would not help to divide both sides by 6. Remember to square both the 6 and the $\sqrt{q-2}$.	$\left(6\sqrt{q-2}\right)^2=\left(\overset{a\,-\,b}{3q-6}\right)^2$ $\overset{a^2\;-\;2ab\;+\;b^2}{}$ $6^2\left(\sqrt{q-2}\right)^2=(3q)^2-2\cdot 3q\cdot 6+6^2$
Simplify, then solve the new equation.	$36(q-2)=9q^2-36q+36$

Distribute.	$36q - 72 = 9q^2 - 36q + 36$
It is a quadratic equation, so get zero on one side.	$0 = 9q^2 - 72q + 108$
Factor the right side.	$0 = 9(q^2 - 8q + 12)$ $0 = 9(q - 6)(q - 2)$
Use the Zero Product Property.	$q - 6 = 0 \quad q - 2 = 0$ $q = 6 \quad q = 2$
The checks are left to you.	The solutions are $q = 6$ and $q = 2$.

TRY IT : : 8.129 Solve: $\sqrt{x - 1} + 2 = \sqrt{2x + 6}$

TRY IT : : 8.130 Solve: $\sqrt{x} + 2 = \sqrt{3x + 4}$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

HOW TO : : USE A PROBLEM SOLVING STRATEGY FOR APPLICATIONS WITH FORMULAS.

Step 1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.

Step 2. **Identify** what we are looking for.

Step 3. **Name** what we are looking for by choosing a variable to represent it.

Step 4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

Step 5. **Solve the equation** using good algebra techniques.

Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of gravity on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t = \frac{\sqrt{h}}{4}.$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

	$t = \frac{\sqrt{h}}{4}$
	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

EXAMPLE 8.66

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the time it takes for the sunglasses to reach the river
Step 3. Name what we are looking.	Let $t =$ time.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$t = \frac{\sqrt{h}}{4}$, and $h = 400$ $t = \frac{\sqrt{400}}{4}$
Step 5. Solve the equation.	$t = \frac{20}{4}$
	$t = 5$
Step 6. Check the answer in the problem and make sure it makes sense.	$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5 \checkmark$
Does 5 seconds seem like a reasonable length of time?	Yes.
Step 7. Answer the question.	It will take 5 seconds for the sunglasses to reach the river.

TRY IT : : 8.131

A helicopter dropped a rescue package from a height of 1,296 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

TRY IT : : 8.132

A window washer dropped a squeegee from a platform 196 feet above the sidewalk Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the speed, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula

$$s = \sqrt{24d}$$

EXAMPLE 8.67

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

Step 1. Read the problem	
Step 2. Identify what we are looking for.	the speed of a car
Step 3. Name what weare looking for,	Let $s =$ the speed.
Step 4. Translate into an equation by writing the appropriate formula. Substitute in the given information.	$s = \sqrt{24d}$, and $d = 190$ $s = \sqrt{24(190)}$
Step 5. Solve the equation.	$s = \sqrt{4{,}560}$
	$s = 67.52777...$
Round to 1 decimal place.	$s \approx 67.5$
	$67.5 \stackrel{?}{\approx} \sqrt{24(190)}$ $67.5 \stackrel{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277... \checkmark$
	The speed of the car before the brakes were applied was 67.5 miles per hour.

TRY IT : : 8.133

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

TRY IT : : 8.134

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

MEDIA : :

Access these online resources for additional instruction and practice with solving radical equations.

- **Solving an Equation Involving a Single Radical (https://openstax.org/l/37RadEquat1)**
- **Solving Equations with Radicals and Rational Exponents (https://openstax.org/l/37RadEquat2)**
- **Solving Radical Equations (https://openstax.org/l/37RadEquat3)**
- **Solve Radical Equations (https://openstax.org/l/37RadEquat4)**
- **Radical Equation Application (https://openstax.org/l/37RadEquat5)**

8.6 EXERCISES

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.

287. $\sqrt{5x-6}=8$

288. $\sqrt{4x-3}=7$

289. $\sqrt{5x+1}=-3$

290. $\sqrt{3y-4}=-2$

291. $\sqrt[3]{2x}=-2$

292. $\sqrt[3]{4x-1}=3$

293. $\sqrt{2m-3}-5=0$

294. $\sqrt{2n-1}-3=0$

295. $\sqrt{6v-2}-10=0$

296. $\sqrt{12u+1}-11=0$

297. $\sqrt{4m+2}+2=6$

298. $\sqrt{6n+1}+4=8$

299. $\sqrt{2u-3}+2=0$

300. $\sqrt{5v-2}+5=0$

301. $\sqrt{u-3}-3=u$

302. $\sqrt{v-10}+10=v$

303. $\sqrt{r-1}=r-1$

304. $\sqrt{s-8}=s-8$

305. $\sqrt[3]{6x+4}=4$

306. $\sqrt[3]{11x+4}=5$

307. $\sqrt[3]{4x+5}-2=-5$

308. $\sqrt[3]{9x-1}-1=-5$

309. $(6x+1)^{\frac{1}{2}}-3=4$

310. $(3x-2)^{\frac{1}{2}}+1=6$

311. $(8x+5)^{\frac{1}{3}}+2=-1$

312. $(12x-5)^{\frac{1}{3}}+8=3$

313. $(12x-3)^{\frac{1}{4}}-5=-2$

314. $(5x-4)^{\frac{1}{4}}+7=9$

315. $\sqrt{x+1}-x+1=0$

316. $\sqrt{y+4}-y+2=0$

317. $\sqrt{z+100}-z=-10$

318. $\sqrt{w+25}-w=-5$

319. $3\sqrt{2x-3}-20=7$

320. $2\sqrt{5x+1}-8=0$

321. $2\sqrt{8r+1}-8=2$

322. $3\sqrt{7y+1}-10=8$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

323. $\sqrt{3u+7}=\sqrt{5u+1}$

324. $\sqrt{4v+1}=\sqrt{3v+3}$

325. $\sqrt{8+2r}=\sqrt{3r+10}$

326. $\sqrt{10+2c}=\sqrt{4c+16}$

327. $\sqrt[3]{5x-1}=\sqrt[3]{x+3}$

328. $\sqrt[3]{8x-5}=\sqrt[3]{3x+5}$

329. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

330. $\sqrt[3]{x^2-x+18}=\sqrt[3]{2x^2-3x-6}$

331. $\sqrt{a}+2=\sqrt{a+4}$

332. $\sqrt{r}+6=\sqrt{r+8}$

333. $\sqrt{u}+1=\sqrt{u+4}$

334. $\sqrt{x}+1=\sqrt{x+2}$

335. $\sqrt{a+5}-\sqrt{a}=1$

336. $-2=\sqrt{d-20}-\sqrt{d}$

337. $\sqrt{2x+1}=1+\sqrt{x}$

338. $\sqrt{3x+1}=1+\sqrt{2x-1}$

339. $\sqrt{2x-1}-\sqrt{x-1}=1$

340. $\sqrt{x+1}-\sqrt{x-2}=1$

341. $\sqrt{x+7} - \sqrt{x-5} = 2$

342. $\sqrt{x+5} - \sqrt{x-3} = 2$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

343. Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

344. Landscaping Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.

345. Gravity A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.

346. Gravity A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t = \frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.

347. Accident investigation The skid marks for a car involved in an accident measured 216 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

348. Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Writing Exercises

349. Explain why an equation of the form $\sqrt{x} + 1 = 0$ has no solution.

350.

ⓐ Solve the equation $\sqrt{r+4} - r + 2 = 0.$

ⓑ Explain why one of the “solutions” that was found was not actually a solution to the equation.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can…	Confidently	With some help	No-I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

CHAPTER 8 REVIEW

KEY TERMS

complex conjugate pair A complex conjugate pair is of the form *a* + *bi*, *a* – *bi*.

complex number A complex number is of the form *a* + *bi*, where *a* and *b* are real numbers. We call *a* the real part and *b* the imaginary part.

complex number system The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit The imaginary unit i is the number whose square is –1. $i^2 = -1$ or $i = \sqrt{-1}.$

like radicals Like radicals are radical expressions with the same index and the same radicand.

radical equation An equation in which a variable is in the radicand of a radical expression is called a radical equation.

radical function A radical function is a function that is defined by a radical expression.

rationalizing the denominator Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

square of a number If $n^2 = m$, then *m* is the square of *n*.

square root of a number If $n^2 = m$, then *n* is a square root of *m*.

standard form A complex number is in standard form when written as $a + bi,$ where *a, b* are real numbers.

KEY CONCEPTS

8.1 Simplify Expressions with Roots

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of *m*'
  - If $n^2 = m$, then $n = \sqrt{m},$ for $n \geq 0.$

  - The square root of *m*, $\sqrt{m},$ is a positive number whose square is *m*.
- ***n*th Root of a Number**
  - If $b^n = a,$ then *b* is an n^{th} root of *a*.
  - The principal n^{th} root of *a* is written $\sqrt[n]{a}.$
  - *n* is called the *index* of the radical.
- **Properties of $\sqrt[n]{a}$**
  - When *n* is an even number and
    - $a \geq 0,$ then $\sqrt[n]{a}$ is a real number
    - $a < 0,$ then $\sqrt[n]{a}$ is not a real number
  - When *n* is an odd number, $\sqrt[n]{a}$ is a real number for all values of *a*.
- **Simplifying Odd and Even Roots**
  - For any integer $n \geq 2,$
    - when *n* is odd $\sqrt[n]{a^n} = a$
    - when *n* is even $\sqrt[n]{a^n} = |a|$
  - We must use the absolute value signs when we take an even root of an expression with a variable in the radical.

8.2 Simplify Radical Expressions

- **Simplified Radical Expression**

- For real numbers *a*, *m* and $n \geq 2$
  $\sqrt[n]{a}$ is considered simplified if *a* has no factors of m^n

- **Product Property of n^th Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$
    $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ and $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
- **How to simplify a radical expression using the Product Property**
  Step 1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
  Step 2. Use the product rule to rewrite the radical as the product of two radicals.
  Step 3. Simplify the root of the perfect power.
- **Quotient Property of Radical Expressions**
  - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,
    $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ and $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$
- **How to simplify a radical expression using the Quotient Property.**
  Step 1. Simplify the fraction in the radicand, if possible.
  Step 2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
  Step 3. Simplify the radicals in the numerator and the denominator.

8.3 Simplify Rational Exponents

- **Rational Exponent** $\boldsymbol{a^{\frac{1}{n}}}$
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2,$ then $a^{\frac{1}{n}} = \sqrt[n]{a}.$
- **Rational Exponent** $\boldsymbol{a^{\frac{m}{n}}}$
  - For any positive integers *m* and *n*,
    $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If *a, b* are real numbers and *m, n* are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{m \cdot n}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, \ a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, \quad a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \ b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, \ a \neq 0$

8.4 Add, Subtract, and Multiply Radical Expressions

- **Product Property of Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n \geq 2$

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \text{ and } \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

- **Special Products**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a+b)(a-b) = a^2 - b^2$
$(a-b)^2 = a^2 - 2ab + b^2$	

8.5 Divide Radical Expressions

- **Quotient Property of Radical Expressions**
    - If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0,$ and for any integer $n \geq 2$ then,

      $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \text{ and } \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

- **Simplified Radical Expressions**
    - A radical expression is considered simplified if there are:
        - no factors in the radicand that have perfect powers of the index
        - no fractions in the radicand
        - no radicals in the denominator of a fraction

8.6 Solve Radical Equations

- **Binomial Squares**

  $(a+b)^2 = a^2 + 2ab + b^2$

  $(a-b)^2 = a^2 - 2ab + b^2$

- **Solve a Radical Equation**

  Step 1. Isolate one of the radical terms on one side of the equation.

  Step 2. Raise both sides of the equation to the power of the index.

  Step 3. Are there any more radicals?
  If yes, repeat Step 1 and Step 2 again.
  If no, solve the new equation.

  Step 4. Check the answer in the original equation.

- **Problem Solving Strategy for Applications with Formulas**

  Step 1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.

  Step 2. Identify what we are looking for.

  Step 3. Name what we are looking for by choosing a variable to represent it.

  Step 4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.

  Step 5. Solve the equation using good algebra techniques.

  Step 6. Check the answer in the problem and make sure it makes sense.

  Step 7. Answer the question with a complete sentence.

- **Falling Objects**
    - On Earth, if an object is dropped from a height of *h* feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.

- **Skid Marks and Speed of a Car**
    - If the length of the skid marks is *d* feet, then the speed, *s*, of the car before the brakes were applied can be found by using the formula $s = \sqrt{24d}$.

8.7 Use Radicals in Functions

- **Properties of $\sqrt[n]{a}$**
    - When *n* is an **even** number and:
      $a \geq 0$, then $\sqrt[n]{a}$ is a real number.
      $a < 0$, then $\sqrt[n]{a}$ is not a real number.
    - When *n* is an **odd** number, $\sqrt[n]{a}$ is a real number for all values of *a*.
- **Domain of a Radical Function**
    - When the **index** of the radical is **even**, the radicand must be greater than or equal to zero.
    - When the **index** of the radical is **odd**, the radicand can be any real number.

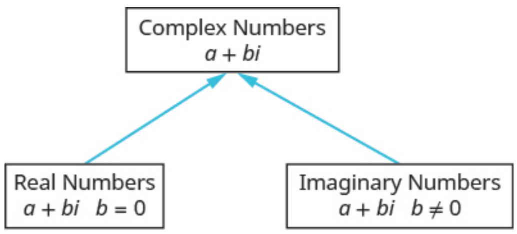
8.8 Use the Complex Number System

- **Square Root of a Negative Number**
    - If *b* is a positive real number, then $\sqrt{-b} = \sqrt{b}i$

	$a + bi$	
$b = 0$	$a + 0 \cdot i$ a	Real number
$b \neq 0$	$a + bi$	Imaginary number
$a = 0$	$0 + bi$ bi	Pure imaginary number

Table 8.32

- A complex number is in **standard form** when written as *a* + *bi*, where *a, b* are real numbers.



- **Product of Complex Conjugates**
    - If *a, b* are real numbers, then
      $(a - bi)(a + bi) = a^2 + b^2$
- **How to Divide Complex Numbers**
  Step 1. Write both the numerator and denominator in standard form.
  Step 2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  Step 3. Simplify and write the result in standard form.

REVIEW EXERCISES

8.1 Simplify Expressions with Roots

Simplify Expressions with Roots

In the following exercises, simplify.

481. ⓐ $\sqrt{225}$ ⓑ $-\sqrt{16}$

482. ⓐ $-\sqrt{169}$ ⓑ $\sqrt{-8}$

483. ⓐ $\sqrt[3]{8}$ ⓑ $\sqrt[4]{81}$ ⓒ $\sqrt[5]{243}$

484. ⓐ $\sqrt[3]{-512}$ ⓑ $\sqrt[4]{-81}$ ⓒ $\sqrt[5]{-1}$

Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

485. ⓐ $\sqrt{68}$ ⓑ $\sqrt[3]{84}$

In the following exercises, approximate each root and round to two decimal places.

486. ⓐ $\sqrt{37}$ ⓑ $\sqrt[3]{84}$ ⓒ $\sqrt[4]{125}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

487.
ⓐ $\sqrt[3]{a^3}$
ⓑ $\sqrt[7]{b^7}$

488.
ⓐ $\sqrt{a^{14}}$
ⓑ $\sqrt{w^{24}}$

489.
ⓐ $\sqrt[4]{m^8}$
ⓑ $\sqrt[5]{n^{20}}$

490.
ⓐ $\sqrt{121m^{20}}$
ⓑ $-\sqrt{64a^2}$

491.
ⓐ $\sqrt[3]{216a^6}$
ⓑ $\sqrt[5]{32b^{20}}$

492.
ⓐ $\sqrt{144x^2y^2}$
ⓑ $\sqrt{169w^8y^{10}}$
ⓒ $\sqrt[3]{8a^{51}b^6}$

8.2 Simplify Radical Expressions

Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

493. $\sqrt{125}$

494. $\sqrt{675}$

495. ⓐ $\sqrt[3]{625}$ ⓑ $\sqrt[6]{128}$

In the following exercises, simplify using absolute value signs as needed.

496.
ⓐ $\sqrt{a^{23}}$
ⓑ $\sqrt[3]{b^8}$
ⓒ $\sqrt[8]{c^{13}}$

497.
ⓐ $\sqrt{80s^{15}}$
ⓑ $\sqrt[5]{96a^7}$
ⓒ $\sqrt[6]{128b^7}$

498.
ⓐ $\sqrt{96r^3s^3}$
ⓑ $\sqrt[3]{80x^7y^6}$
ⓒ $\sqrt[4]{80x^8y^9}$

499.
ⓐ $\sqrt[5]{-32}$
ⓑ $\sqrt[8]{-1}$

500.
ⓐ $8+\sqrt{96}$
ⓑ $\dfrac{2+\sqrt{40}}{2}$

Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

501. ⓐ $\sqrt{\frac{72}{98}}$ ⓑ $\sqrt[3]{\frac{24}{81}}$ ⓒ $\sqrt[4]{\frac{6}{96}}$

502. ⓐ $\sqrt{\frac{y^4}{y^8}}$ ⓑ $\sqrt[5]{\frac{u^{21}}{u^{11}}}$ ⓒ $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

503. $\sqrt{\frac{300m^5}{64}}$

504.

ⓐ $\sqrt{\frac{28p^7}{q^2}}$

ⓑ $\sqrt[3]{\frac{81s^8}{t^3}}$

ⓒ $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

505.

ⓐ $\sqrt{\frac{27p^2q}{108p^4q^3}}$

ⓑ $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$

ⓒ $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

506.

ⓐ $\frac{\sqrt{80q^5}}{\sqrt{5q}}$

ⓑ $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$

ⓒ $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

8.3 Simplify Rational Exponents

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

507. ⓐ $r^{\frac{1}{2}}$ ⓑ $s^{\frac{1}{3}}$ ⓒ $t^{\frac{1}{4}}$

In the following exercises, write with a rational exponent.

508. ⓐ $\sqrt{21p}$ ⓑ $\sqrt[4]{8q}$ ⓒ $4\sqrt[6]{36r}$

In the following exercises, simplify.

509.

ⓐ $625^{\frac{1}{4}}$

ⓑ $243^{\frac{1}{5}}$

ⓒ $32^{\frac{1}{5}}$

510.

ⓐ $(-1{,}000)^{\frac{1}{3}}$

ⓑ $-1{,}000^{\frac{1}{3}}$

ⓒ $(1{,}000)^{-\frac{1}{3}}$

511.

ⓐ $(-32)^{\frac{1}{5}}$

ⓑ $(243)^{-\frac{1}{5}}$

ⓒ $-125^{\frac{1}{3}}$

Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

512.

ⓐ $\sqrt[4]{r^7}$

ⓑ $\left(\sqrt[5]{2pq}\right)^3$

ⓒ $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

In the following exercises, simplify.

513.

ⓐ $25^{\frac{3}{2}}$

ⓑ $9^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{2}{3}}$

514.

ⓐ $-64^{\frac{3}{2}}$

ⓑ $-64^{-\frac{3}{2}}$

ⓒ $(-64)^{\frac{3}{2}}$

Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

515.

ⓐ $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$

ⓑ $\left(b^{15}\right)^{\frac{3}{5}}$

ⓒ $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$

516.

ⓐ $\frac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$

ⓑ $\left(\frac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

8.4 Add, Subtract and Multiply Radical Expressions

Add and Subtract Radical Expressions

In the following exercises, simplify.

517.

ⓐ $7\sqrt{2} - 3\sqrt{2}$

ⓑ $7\sqrt[3]{p} + 2\sqrt[3]{p}$

ⓒ $5\sqrt[3]{x} - 3\sqrt[3]{x}$

518.

ⓐ $\sqrt{11b} - 5\sqrt{11b} + 3\sqrt{11b}$

ⓑ $8\sqrt[4]{11cd} + 5\sqrt[4]{11cd} - 9\sqrt[4]{11cd}$

519.

ⓐ $\sqrt{48} + \sqrt{27}$

ⓑ $\sqrt[3]{54} + \sqrt[3]{128}$

ⓒ $6\sqrt[4]{5} - \frac{3}{2}\sqrt[4]{320}$

520.

ⓐ $\sqrt{80c^7} - \sqrt{20c^7}$

ⓑ $2\sqrt[4]{162r^{10}} + 4\sqrt[4]{32r^{10}}$

521. $3\sqrt{75y^2} + 8y\sqrt{48} - \sqrt{300y^2}$

Multiply Radical Expressions

In the following exercises, simplify.

522.

ⓐ $(5\sqrt{6})(-\sqrt{12})$

ⓑ $\left(-2\sqrt[4]{18}\right)\left(-\sqrt[4]{9}\right)$

523.

ⓐ $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

ⓑ $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

524.

ⓐ $\sqrt{11}(8 + 4\sqrt{11})$

ⓑ $\sqrt[3]{3}\left(\sqrt[3]{9} + \sqrt[3]{18}\right)$

525.

ⓐ $(3 - 2\sqrt{7})(5 - 4\sqrt{7})$

ⓑ $\left(\sqrt[3]{x} - 5\right)\left(\sqrt[3]{x} - 3\right)$

526. $\left(2\sqrt{7} - 5\sqrt{11}\right)\left(4\sqrt{7} + 9\sqrt{11}\right)$

527.
ⓐ $\left(4+\sqrt{11}\right)^2$
ⓑ $\left(3-2\sqrt{5}\right)^2$

528. $\left(7+\sqrt{10}\right)\left(7-\sqrt{10}\right)$

529. $\left(\sqrt[3]{3x}+2\right)\left(\sqrt[3]{3x}-2\right)$

8.5 Divide Radical Expressions

Divide Square Roots

In the following exercises, simplify.

530.
ⓐ $\frac{\sqrt{48}}{\sqrt{75}}$
ⓑ $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$

531.
ⓐ $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$
ⓑ $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

532. ⓐ $\frac{8}{\sqrt{3}}$ ⓑ $\sqrt{\frac{7}{40}}$ ⓒ $\frac{8}{\sqrt{2y}}$

533. ⓐ $\frac{1}{\sqrt[3]{11}}$ ⓑ $\sqrt[3]{\frac{7}{54}}$ ⓒ $\frac{3}{\sqrt[3]{3x^2}}$

534. ⓐ $\frac{1}{\sqrt[4]{4}}$ ⓑ $\sqrt[4]{\frac{9}{32}}$ ⓒ $\frac{6}{\sqrt[4]{9x^3}}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

535. $\frac{7}{2-\sqrt{6}}$

536. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

537. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

8.6 Solve Radical Equations

Solve Radical Equations

In the following exercises, solve.

538. $\sqrt{4x-3}=7$

539. $\sqrt{5x+1}=-3$

540. $\sqrt[3]{4x-1}=3$

541. $\sqrt{u-3}+3=u$

542. $\sqrt[3]{4x+5}-2=-5$

543. $(8x+5)^{\frac{1}{3}}+2=-1$

544. $\sqrt{y+4}-y+2=0$

545. $2\sqrt{8r+1}-8=2$

Solve Radical Equations with Two Radicals

In the following exercises, solve.

546. $\sqrt{10+2c}=\sqrt{4c+16}$

547.
$\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$

548. $\sqrt{r}+6=\sqrt{r+8}$

549. $\sqrt{x+1}-\sqrt{x-2}=1$

Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

550. Landscaping Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s = \sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.

551. Accident investigation An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

8.7 Use Radicals in Functions

Evaluate a Radical Function

In the following exercises, evaluate each function.

552. $g(x) = \sqrt{6x+1}$, find

ⓐ $g(4)$

ⓑ $g(8)$

553. $G(x) = \sqrt{5x-1}$, find

ⓐ $G(5)$

ⓑ $G(2)$

554. $h(x) = \sqrt[3]{x^2-4}$, find

ⓐ $h(-2)$

ⓑ $h(6)$

555. For the function $g(x) = \sqrt[4]{4-4x}$, find

ⓐ $g(1)$

ⓑ $g(-3)$

Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

556. $g(x) = \sqrt{2-3x}$

557. $F(x) = \sqrt{\dfrac{x+3}{x-2}}$

558. $f(x) = \sqrt[3]{4x^2-16}$

559. $F(x) = \sqrt[4]{10-7x}$

Graph Radical Functions

In the following exercises, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

560. $g(x) = \sqrt{x+4}$

561. $g(x) = 2\sqrt{x}$

562. $f(x) = \sqrt[3]{x-1}$

563. $f(x) = \sqrt[3]{x}+3$

8.8 Use the Complex Number System

Evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

564.

ⓐ $\sqrt{-100}$

ⓑ $\sqrt{-13}$

ⓒ $\sqrt{-45}$

Add or Subtract Complex Numbers

In the following exercises, add or subtract.

565. $\sqrt{-50}+\sqrt{-18}$

566. $(8-i)+(6+3i)$

567. $(6+i)-(-2-4i)$

568. $\left(-7-\sqrt{-50}\right)-\left(-32-\sqrt{-18}\right)$

Multiply Complex Numbers

In the following exercises, multiply.

569. $(-2-5i)(-4+3i)$

570. $-6i(-3-2i)$

571. $\sqrt{-4}\cdot\sqrt{-16}$

572. $\left(5-\sqrt{-12}\right)\left(-3+\sqrt{-75}\right)$

In the following exercises, multiply using the Product of Binomial Squares Pattern.

573. $(-2-3i)^2$

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

574. $(9-2i)(9+2i)$

Divide Complex Numbers

In the following exercises, divide.

575. $\dfrac{2+i}{3-4i}$

576. $\dfrac{-4}{3-2i}$

Simplify Powers of *i*

In the following exercises, simplify.

577. i^{48}

578. i^{255}

PRACTICE TEST

In the following exercises, simplify using absolute values as necessary.

579. $\sqrt[3]{125x^9}$

580. $\sqrt{169x^8y^6}$

581. $\sqrt[3]{72x^8y^4}$

582. $\sqrt{\dfrac{45x^3y^4}{180x^5y^2}}$

In the following exercises, simplify. Assume all variables are positive.

583. ⓐ $216^{-\frac{1}{4}}$ ⓑ $-49^{\frac{3}{2}}$

584. $\sqrt{-45}$

585. $\dfrac{x^{-\frac{1}{4}} \cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$

586. $\left(\dfrac{8x^{\frac{2}{3}}y^{-\frac{5}{2}}}{x^{-\frac{7}{3}}y^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

587. $\sqrt{48x^5} - \sqrt{75x^5}$

588. $\sqrt{27x^2} - 4x\sqrt{12} + \sqrt{108x^2}$

589. $2\sqrt{12x^5} \cdot 3\sqrt{6x^3}$

590. $\sqrt[3]{4}\left(\sqrt[3]{16} - \sqrt[3]{6}\right)$

591. $(4 - 3\sqrt{3})(5 + 2\sqrt{3})$

592. $\dfrac{\sqrt[3]{128}}{\sqrt[3]{54}}$

593. $\dfrac{\sqrt{245xy^{-4}}}{\sqrt{45x^{-4}y^3}}$

594. $\dfrac{1}{\sqrt[3]{5}}$

595. $\dfrac{3}{2 + \sqrt{3}}$

596. $\sqrt{-4} \cdot \sqrt{-9}$

597. $-4i(-2 - 3i)$

598. $\dfrac{4 + i}{3 - 2i}$

599. i^{172}

In the following exercises, solve.

600. $\sqrt{2x + 5} + 8 = 6$

601. $\sqrt{x + 5} + 1 = x$

602. $\sqrt[3]{2x^2 - 6x - 23} = \sqrt[3]{x^2 - 3x + 5}$

In the following exercise, ⓐ find the domain of the function ⓑ graph the function ⓒ use the graph to determine the range.

603. $g(x) = \sqrt{x + 2}$

9 QUADRATIC EQUATIONS AND FUNCTIONS

Figure 9.1 Several companies have patented contact lenses equipped with cameras, suggesting that they may be the future of wearable camera technology. (credit: “intographics”/Pixabay)

Chapter Outline

- **9.1** Solve Quadratic Equations Using the Square Root Property
- **9.2** Solve Quadratic Equations by Completing the Square
- **9.3** Solve Quadratic Equations Using the Quadratic Formula
- **9.4** Solve Quadratic Equations in Quadratic Form
- **9.5** Solve Applications of Quadratic Equations
- **9.6** Graph Quadratic Functions Using Properties
- **9.7** Graph Quadratic Functions Using Transformations
- **9.8** Solve Quadratic Inequalities

Introduction

Blink your eyes. You’ve taken a photo. That’s what will happen if you are wearing a contact lens with a built-in camera. Some of the same technology used to help doctors see inside the eye may someday be used to make cameras and other devices. These technologies are being developed by biomedical engineers using many mathematical principles, including an understanding of quadratic equations and functions. In this chapter, you will explore these kinds of equations and learn to solve them in different ways. Then you will solve applications modeled by quadratics, graph them, and extend your understanding to quadratic inequalities.

9.1 Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x - h)^2 = k$ using the Square Root Property

Be Prepared!

Before you get started, take this readiness quiz.

1. Simplify: $\sqrt{128}$.
   If you missed this problem, review **Example 8.13**.
2. Simplify: $\sqrt{\frac{32}{5}}$.

If you missed this problem, review **Example 8.50**.

3. Factor: $9x^2 - 12x + 4$.
   If you missed this problem, review **Example 6.23**.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the diffe ence of squares.	$(x-3)(x+3) = 0$
Use the Zero Product Property.	$x - 3 = 0 \quad x - 3 = 0$
Solve each equation.	$x = 3 \quad x = -3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4,\ x = -4$ and $x = 5,\ x = -5$.

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so −13 is also a square root of 169. Therefore, both 13 and −13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

$$\text{If } n^2 = m, \text{ then } n \text{ is a square root of } m.$$

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad \text{or} \quad x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm\sqrt{9}$
	$x = \pm 3$
	So $x = 3$ or $x = -3$.

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

	$x^2 = 7$
Use the Square Root Property.	$x = \sqrt{7}, \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

EXAMPLE 9.1 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $AX^2 = K$ USING THE SQUARE ROOT PROPERTY

Solve: $x^2 - 50 = 0$.

Solution

Step 1. Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 - 50 = 0$ $x^2 = 50$
Step 2. Use Square Root Property.	Remember to write the ± symbol.	$x = \pm\sqrt{50}$
Step 3. Simplify the radical.	Rewrite to show two solutions.	$x = \pm\sqrt{25} \cdot \sqrt{2}$ $x = \pm 5\sqrt{2}$ $x = 5\sqrt{2},\ x = -5\sqrt{2}$
Step 4. Check the solutions.	Substitute in $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$	$x^2 - 50 = 0$ $(5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ ✓ $x^2 - 50 = 0$ $(-5\sqrt{2})^2 - 50 \stackrel{?}{=} 0$ $25 \cdot 2 - 50 \stackrel{?}{=} 0$ $0 = 0$ ✓

TRY IT : : 9.1 Solve: $x^2 - 48 = 0$.

TRY IT : : 9.2 Solve: $y^2 - 27 = 0$.

The steps to take to use the Square Root Property to solve a quadratic equation are listed here.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY.

Step 1. Isolate the quadratic term and make its coefficient one.
Step 2. Use Square Root Property.
Step 3. Simplify the radical.
Step 4. Check the solutions.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

EXAMPLE 9.2

Solve: $3z^2 = 108$.

Solution

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6, \quad z = -6$
Check the solutions: $3z^2 = 108$ $\quad$ $3z^2 = 108$ $3(6)^2 \stackrel{?}{=} 108$ $\quad$ $3(-6)^2 \stackrel{?}{=} 108$ $3(36) \stackrel{?}{=} 108$ $\quad$ $3(36) \stackrel{?}{=} 108$ $108 = 108\ \checkmark$ $\quad$ $108 = 108\ \checkmark$	

TRY IT : : 9.3 Solve: $2x^2 = 98$.

TRY IT : : 9.4 Solve: $5m^2 = 80$.

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

EXAMPLE 9.3

Solve: $x^2 + 72 = 0$.

Solution

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}\, i$
Simplify the radical.	$x = \pm 6\sqrt{2}\, i$
Rewrite to show two solutions.	$x = 6\sqrt{2}\, i, \quad x = -6\sqrt{2}\, i$

Check the solutions:

$$x^2 + 72 = 0 \qquad x^2 + 72 = 0$$
$$\left(6\sqrt{2}\,i\right)^2 + 72 \stackrel{?}{=} 0 \qquad \left(6\sqrt{2}\,i\right)^2 + 72 \stackrel{?}{=} 0$$
$$6^2\left(\sqrt{2}\right)^2 i^2 + 72 \stackrel{?}{=} 0 \qquad (-6)^2\left(\sqrt{2}\right)^2 i^2 + 72 \stackrel{?}{=} 0$$
$$36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0 \qquad 36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$$
$$0 = 0 \checkmark \qquad 0 = 0 \checkmark$$

> **TRY IT : :** 9.5 Solve: $c^2 + 12 = 0$.

> **TRY IT : :** 9.6 Solve: $q^2 + 24 = 0$.

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

EXAMPLE 9.4

Solve: $\frac{2}{3}u^2 + 5 = 17$.

Solution

	$\frac{2}{3}u^2 + 5 = 17$
Isolate the quadratic term.	$\frac{2}{3}u^2 = 12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$
Simplify.	$u^2 = 18$
Use the Square Root Property.	$u = \pm\sqrt{18}$
Simplify the radical.	$u = \pm\sqrt{9 \cdot 2}$
Simplify.	$u = \pm 3\sqrt{2}$
Rewrite to show two solutions.	$u = 3\sqrt{2}, \quad u = -3\sqrt{2}$

Check:

$$\frac{2}{3}u^2 + 5 = 17 \qquad \frac{2}{3}u^2 + 5 = 17$$
$$\frac{2}{3}\left(3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17 \qquad \frac{2}{3}\left(-3\sqrt{2}\right)^2 + 5 \stackrel{?}{=} 17$$
$$\frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17 \qquad \frac{2}{3} \cdot 18 + 5 \stackrel{?}{=} 17$$
$$12 + 5 \stackrel{?}{=} 17 \qquad 12 + 5 \stackrel{?}{=} 17$$
$$17 = 17 \checkmark \qquad 17 = 17 \checkmark$$

> **TRY IT : :** 9.7 Solve: $\frac{1}{2}x^2 + 4 = 24$.

TRY IT : : 9.8 Solve: $\frac{3}{4}y^2 - 3 = 18$.

The solutions to some equations may have fractions inside the radicals. When this happens, we must rationalize the denominator.

EXAMPLE 9.5

Solve: $2x^2 - 8 = 41$.

Solution

	$2x^2 - 8 = 41$
Isolate the quadratic term.	$2x^2 = 49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2} = \frac{49}{2}$
Simplify.	$x^2 = \frac{49}{2}$
Use the Square Root Property.	$x = \pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x = \pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x = \pm\frac{\sqrt{49} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$
Simplify.	$x = \pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x = \frac{7\sqrt{2}}{2},\quad x = -\frac{7\sqrt{2}}{2}$
Check: We leave the check for you.	

TRY IT : : 9.9 Solve: $5r^2 - 2 = 34$.

TRY IT : : 9.10 Solve: $3t^2 + 6 = 70$.

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

We can use the Square Root Property to solve an equation of the form $a(x - h)^2 = k$ as well. Notice that the quadratic term, x, in the original form $ax^2 = k$ is replaced with $(x - h)$.

$$ax^2 = k \qquad a(x - h)^2 = k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x - h)^2$.

EXAMPLE 9.6

Solve: $4(y - 7)^2 = 48$.

Solution

	$4(y-7)^2 = 48$
Divide both sides by the coefficient 4.	$(y-7)^2 = 12$
Use the Square Root Property on the binomial	$y - 7 = \pm\sqrt{12}$
Simplify the radical.	$y - 7 = \pm 2\sqrt{3}$
Solve for y.	$y = 7 \pm 2\sqrt{3}$
Rewrite to show two solutions.	$y = 7 + 2\sqrt{3}, \quad y = 7 - 2\sqrt{3}$

Check:

$4(y-7)^2 = 48$	$4(y-7)^2 = 48$
$4(7+2\sqrt{3}-7)^2 \stackrel{?}{=} 48$	$4(7-2\sqrt{3}-7)^2 \stackrel{?}{=} 48$
$4(2\sqrt{3})^2 \stackrel{?}{=} 48$	$4(-2\sqrt{3})^2 \stackrel{?}{=} 48$
$4(12) \stackrel{?}{=} 48$	$4(12) \stackrel{?}{=} 48$
$48 = 48$ ✓	$48 = 48$ ✓

TRY IT : : 9.11 Solve: $3(a-3)^2 = 54$.

TRY IT : : 9.12 Solve: $2(b+2)^2 = 80$.

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

EXAMPLE 9.7

Solve: $\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$.

Solution

	$\left(x - \frac{1}{3}\right)^2 = \frac{5}{9}$
Use the Square Root Property.	$x - \frac{1}{3} = \pm\sqrt{\frac{5}{9}}$
Rewrite the radical as a fraction of square roots.	$x - \frac{1}{3} = \pm\frac{\sqrt{5}}{\sqrt{9}}$
Simplify the radical.	$x - \frac{1}{3} = \pm\frac{\sqrt{5}}{3}$
Solve for x.	$x = \frac{1}{3} \pm \frac{\sqrt{5}}{3}$
Rewrite to show two solutions.	$x = \frac{1}{3} + \frac{\sqrt{5}}{3},\quad x = \frac{1}{3} - \frac{\sqrt{5}}{3}$

Check:
We leave the check for you.

TRY IT : : 9.13 Solve: $\left(x - \frac{1}{2}\right)^2 = \frac{5}{4}$.

TRY IT : : 9.14 Solve: $\left(y + \frac{3}{4}\right)^2 = \frac{7}{16}$.

We will start the solution to the next example by isolating the binomial term.

EXAMPLE 9.8

Solve: $2(x - 2)^2 + 3 = 57$.

Solution

	$2(x - 2)^2 + 3 = 57$
Subtract 3 from both sides to isolate the binomial term.	$2(x - 2)^2 = 54$
Divide both sides by 2.	$(x - 2)^2 = 27$
Use the Square Root Property.	$x - 2 = \pm\sqrt{27}$
Simplify the radical.	$x - 2 = \pm 3\sqrt{3}$
Solve for x.	$x = 2 \pm 3\sqrt{3}$
Rewrite to show two solutions.	$x = 2 + 3\sqrt{3},\quad x = 2 - 3\sqrt{3}$

Check:
We leave the check for you.

TRY IT : : 9.15 Solve: $5(a - 5)^2 + 4 = 104$.

TRY IT : : 9.16 Solve: $3(b + 3)^2 - 8 = 88$.

Sometimes the solutions are complex numbers.

EXAMPLE 9.9

Solve: $(2x-3)^2 = -12$.

Solution

	$(2x-3)^2 = -12$
Use the Square Root Property.	$2x-3 = \pm\sqrt{-12}$
Simplify the radical.	$2x-3 = \pm 2\sqrt{3}\,i$
Add 3 to both sides.	$2x = 3 \pm 2\sqrt{3}\,i$
Divide both sides by 2.	$x = \dfrac{3 \pm 2\sqrt{3}\,i}{2}$
Rewrite in standard form.	$x = \dfrac{3}{2} \pm \dfrac{2\sqrt{3}\,i}{2}$
Simplify.	$x = \dfrac{3}{2} \pm \sqrt{3}\,i$
Rewrite to show two solutions.	$x = \dfrac{3}{2} + \sqrt{3}\,i, \quad x = \dfrac{3}{2} - \sqrt{3}\,i$

Check:
We leave the check for you.

TRY IT : : 9.17 Solve: $(3r+4)^2 = -8$.

TRY IT : : 9.18 Solve: $(2t-8)^2 = -10$.

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

EXAMPLE 9.10

Solve: $4n^2 + 4n + 1 = 16$.

Solution

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2 + 4n + 1 = 16$
Factor the perfect square trinomial.	$(2n+1)^2 = 16$
Use the Square Root Property.	$2n+1 = \pm\sqrt{16}$
Simplify the radical.	$2n+1 = \pm 4$
Solve for n.	$2n = -1 \pm 4$
Divide each side by 2.	$\dfrac{2n}{2} = \dfrac{-1 \pm 4}{2}$ $n = \dfrac{-1 \pm 4}{2}$
Rewrite to show two solutions.	$n = \dfrac{-1+4}{2}, \quad n = \dfrac{-1-4}{2}$
Simplify each equation.	$n = \dfrac{3}{2}, \qquad n = -\dfrac{5}{2}$

Check:

$$4n^2 + 4n + 1 = 16 \qquad 4n^2 + 4n + 1 = 16$$

$$4\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(-\frac{5}{2}\right)^2 + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$4\left(\frac{9}{4}\right) + 4\left(\frac{3}{2}\right) + 1 \stackrel{?}{=} 16 \qquad 4\left(\frac{25}{4}\right) + 4\left(-\frac{5}{2}\right) + 1 \stackrel{?}{=} 16$$

$$9 + 6 + 1 \stackrel{?}{=} 16 \qquad 25 - 10 + 1 \stackrel{?}{=} 16$$

$$16 = 16 \checkmark \qquad 16 = 16 \checkmark$$

TRY IT : : 9.19 Solve: $9m^2 - 12m + 4 = 25$.

TRY IT : : 9.20 Solve: $16n^2 + 40n + 25 = 4$.

MEDIA : :

Access this online resource for additional instruction and practice with using the Square Root Property to solve quadratic equations.

- **Solving Quadratic Equations: The Square Root Property (https://openstax.org/l/37SqRtProp1)**
- **Using the Square Root Property to Solve Quadratic Equations (https://openstax.org/l/37SqRtProp2)**

9.1 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2 = 49$

2. $b^2 = 144$

3. $r^2 - 24 = 0$

4. $t^2 - 75 = 0$

5. $u^2 - 300 = 0$

6. $v^2 - 80 = 0$

7. $4m^2 = 36$

8. $3n^2 = 48$

9. $\frac{4}{3}x^2 = 48$

10. $\frac{5}{3}y^2 = 60$

11. $x^2 + 25 = 0$

12. $y^2 + 64 = 0$

13. $x^2 + 63 = 0$

14. $y^2 + 45 = 0$

15. $\frac{4}{3}x^2 + 2 = 110$

16. $\frac{2}{3}y^2 - 8 = -2$

17. $\frac{2}{5}a^2 + 3 = 11$

18. $\frac{3}{2}b^2 - 7 = 41$

19. $7p^2 + 10 = 26$

20. $2q^2 + 5 = 30$

21. $5y^2 - 7 = 25$

22. $3x^2 - 8 = 46$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve each equation.

23. $(u - 6)^2 = 64$

24. $(v + 10)^2 = 121$

25. $(m - 6)^2 = 20$

26. $(n + 5)^2 = 32$

27. $\left(r - \frac{1}{2}\right)^2 = \frac{3}{4}$

28. $\left(x + \frac{1}{5}\right)^2 = \frac{7}{25}$

29. $\left(y + \frac{2}{3}\right)^2 = \frac{8}{81}$

30. $\left(t - \frac{5}{6}\right)^2 = \frac{11}{25}$

31. $(a - 7)^2 + 5 = 55$

32. $(b - 1)^2 - 9 = 39$

33. $4(x + 3)^2 - 5 = 27$

34. $5(x + 3)^2 - 7 = 68$

35. $(5c + 1)^2 = -27$

36. $(8d - 6)^2 = -24$

37. $(4x - 3)^2 + 11 = -17$

38. $(2y + 1)^2 - 5 = -23$

39. $m^2 - 4m + 4 = 8$

40. $n^2 + 8n + 16 = 27$

41. $x^2 - 6x + 9 = 12$

42. $y^2 + 12y + 36 = 32$

43. $25x^2 - 30x + 9 = 36$

44. $9y^2 + 12y + 4 = 9$

45. $36x^2 - 24x + 4 = 81$

46. $64x^2 + 144x + 81 = 25$

Mixed Practice

In the following exercises, solve using the Square Root Property.

47. $2r^2 = 32$

48. $4t^2 = 16$

49. $(a - 4)^2 = 28$

50. $(b + 7)^2 = 8$

51. $9w^2 - 24w + 16 = 1$

52. $4z^2 + 4z + 1 = 49$

53. $a^2 - 18 = 0$

54. $b^2 - 108 = 0$

55. $\left(p - \frac{1}{3}\right)^2 = \frac{7}{9}$

56. $\left(q - \frac{3}{5}\right)^2 = \frac{3}{4}$

57. $m^2 + 12 = 0$

58. $n^2 + 48 = 0.$

59. $u^2 - 14u + 49 = 72$

60. $v^2 + 18v + 81 = 50$

61. $(m - 4)^2 + 3 = 15$

62. $(n - 7)^2 - 8 = 64$

63. $(x + 5)^2 = 4$

64. $(y - 4)^2 = 64$

65. $6c^2 + 4 = 29$

66. $2d^2 - 4 = 77$

67. $(x - 6)^2 + 7 = 3$

68. $(y - 4)^2 + 10 = 9$

Writing Exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x + 2)^2 = 16$.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2 = k$ using the square root property.			
solve quadratic equations of the form $a(x - h)^2 = k$ using the square root property.			

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

ⓑ *If most of your checks were:*

...confidently. *Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.*

...with some help. *This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?*

...no - I don't get it! *This is a warning sign and you must not ignore it. You should get help right away or you will quickly be* *overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.*

9.2 Solve Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square
- Solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square

Be Prepared!

Before you get started, take this readiness quiz.

1. Expand: $(x + 9)^2$.
   If you missed this problem, review Example 5.32.
2. Factor $y^2 - 14y + 49$.
   If you missed this problem, review Example 6.9.
3. Factor $5n^2 + 40n + 80$.
   If you missed this problem, review Example 6.14.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y - 7)^2 = 12$ because the left side was a perfect square.

$$\begin{aligned} (y-7)^2 &= 12 \\ y-7 &= \pm\sqrt{12} \\ y-7 &= \pm 2\sqrt{3} \\ y &= 7 \pm 2\sqrt{3} \end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x - k)^2$ in order to use the Square Root Property.

$$\begin{aligned} x^2 - 10x + 25 &= 18 \\ (x-5)^2 &= 18 \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$$\begin{array}{cc} (x+9)^2 & (y-7)^2 \\ (x+9)(x+9) & (y-7)(y-7) \\ x^2 + 9x + 9x + 81 & y^2 - 7y - 7y + 49 \\ x^2 + 18x + 81 & y^2 - 14y + 49 \end{array}$$

We restate the patterns here for reference.

Binomial Squares Pattern

If *a* and *b* are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

We can use this pattern to “make” a perfect square.

We will start with the expression $x^2 + 6x$. Since there is a plus sign between the two terms, we will use the $(a + b)^2$ pattern, $a^2 + 2ab + b^2 = (a + b)^2$.

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + \_\_}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of $x^2 + 6x$ is a square, x^2. This tells us that $a = x$.

$$\overset{a^2 + \quad 2ab \quad + b^2}{x^2 + 2 \cdot x \cdot b + b^2}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b = 3$.

$$\overset{a^2 + \quad 2ab \quad + b^2}{x^2 + 2 \cdot 3 \cdot x + \_\_}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2 = 9$.

$$\overset{a^2 + 2ab + b^2}{x^2 + 6x + 9}$$

We can now factor.

$$\overset{(a+b)^2}{(x + 3)^2}$$

So we found that adding 9 to $x^2 + 6x$ ‘completes the square’, and we write it as $(x + 3)^2$.

HOW TO : : COMPLETE A SQUARE OF $x^2 + bx$.

Step 1. Identify b, the coefficient of x.

Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared.

EXAMPLE 9.11

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $x^2 - 26x$ ⓑ $y^2 - 9y$ ⓒ $n^2 + \frac{1}{2}n$

Solution

ⓐ

	$x^2 - bx$ $x^2 - 26x$
The coefficient of x is −26.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-26)\right)^2$ $(13)^2$ 169	
Add 169 to the binomial to complete the square.	$x^2 - 26x + 169$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x - 13)^2$

ⓑ

	$x^2 - bx$ $y^2 - 9y$
The coefficient of y is -9.	
Find $\left(\frac{1}{2}b\right)^2$. $\left(\frac{1}{2}\cdot(-9)\right)^2$ $\left(-\frac{9}{2}\right)^2$ $\frac{81}{4}$	
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2 - 9y + \frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(y - \frac{9}{2}\right)^2$

ⓒ

	$x^2 + bx$ $n^2 + \frac{1}{2}n$
The coefficient of n is $\frac{1}{2}$.	

Find $\left(\frac{1}{2}b\right)^2$.

$\left(\frac{1}{2} \cdot \frac{1}{2}\right)^2$

$\left(\frac{1}{4}\right)^2$

$\frac{1}{16}$

Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2 + \frac{1}{2}n + \frac{1}{16}$
Rewrite as a binomial square.	$\left(n + \frac{1}{4}\right)^2$

TRY IT : : 9.21

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $a^2 - 20a$ ⓑ $m^2 - 5m$ ⓒ $p^2 + \frac{1}{4}p$

TRY IT : : 9.22

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

ⓐ $b^2 - 4b$ ⓑ $n^2 + 13n$ ⓒ $q^2 - \frac{2}{3}q$

Solve Quadratic Equations of the Form $x^2 + bx + c = 0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a quadratic equation by completing the square too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2 + 6x = 40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2 + 6x = 40$
	$x^2 + 6x + \_\_ = 40 + \_\_$
	$x^2 + 6x + 9 = 40 + 9$
Add 9 to both sides to complete the square.	$(x + 3)^2 = 49$

Now the equation is in the form to solve using the Square Root Property! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

EXAMPLE 9.12 HOW TO SOLVE A QUADRATIC EQUATION OF THE FORM $x^2 + bx + c = 0$ BY COMPLETING THE SQUARE

Solve by completing the square: $x^2 + 8x = 48$.

Solution

Step 1. Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$\underbrace{x^2 + 8x}_{x^2+bx} = \underbrace{48}_{c}$
Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	Take half of 8 and square it. $4^2 = 16$ Add 16 to BOTH sides of the equation.	$x^2 + 8x + \underset{\left(\frac{1}{2}\cdot 8\right)^2}{\underline{\quad}} = 48$ $x^2 + 8x + 16 = 48 + 16$
Step 3. Factor the perfect square trinomial as a binomial square.	$x^2 + 8x + 16 = (x + 4)^2$ Add the terms on the right.	$(x + 4)^2 = 64$
Step 4. Use the Square Root Property.		$x + 4 = \pm\sqrt{64}$
Step 5. Simplify the radical and then solve the two resulting equations.		$x + 4 = \pm 8$ $x + 4 = 8 \quad x + 4 = -8$ $x = 4 \quad x = -12$
Step 6. Check the solutions.	Put each answer in the original equation to check. Substitute $x = 4$. Substitute $x = -12$.	$x^2 + 8x = 48$ $(4)^2 + 8(4) \stackrel{?}{=} 48$ $16 + 32 \stackrel{?}{=} 48$ $48 = 48$ ✓ $x^2 + 8x = 48$ $(-12)^2 + 8(-12) \stackrel{?}{=} 48$ $144 - 96 \stackrel{?}{=} 48$ $48 = 48$ ✓

TRY IT : : 9.23 Solve by completing the square: $x^2 + 4x = 5$.

TRY IT : : 9.24 Solve by completing the square: $y^2 - 10y = -9$.

The steps to solve a quadratic equation by completing the square are listed here.

HOW TO : : SOLVE A QUADRATIC EQUATION OF THE FORM $x^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Isolate the variable terms on one side and the constant terms on the other.

Step 2. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 4. Use the Square Root Property.

Step 5. Simplify the radical and then solve the two resulting equations.

Step 6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

EXAMPLE 9.13

Solve by completing the square: $x^2 + 4x = -21$.

Solution

	$\overset{x^2\ +\ bx}{x^2 + 4x} = \overset{c}{-21}$
The variable terms are on the left side. Take half of 4 and square it.	$x^2 + 4x + \underset{\left(\frac{1}{2}\cdot 4\right)^2}{\underline{\qquad}} = -21$
$\left(\frac{1}{2}(4)\right)^2 = 4$	
Add 4 to both sides.	$x^2 + 4x + 4 = -21 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+2)^2 = -17$
Use the Square Root Property.	$x + 2 = \pm\sqrt{-17}$
Simplify using complex numbers.	$x + 2 = \pm\sqrt{17}\,i$
Subtract 2 from each side.	$x = -2 \pm \sqrt{17}\,i$
Rewrite to show two solutions.	$x = -2 + \sqrt{17}\,i, \quad x = -2 - \sqrt{17}\,i$
We leave the check to you.	

TRY IT : : 9.25 Solve by completing the square: $y^2 - 10y = -35$.

TRY IT : : 9.26 Solve by completing the square: $z^2 + 8z = -19$.

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

EXAMPLE 9.14

Solve by completing the square: $y^2 - 18y = -6$.

Solution

	$\overset{x^2\ -\ bx}{y^2 - 18y} = \overset{c}{-6}$
The variable terms are on the left side. Take half of -18 and square it.	
$\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + \underset{\left(\frac{1}{2}\cdot(-18)\right)^2}{\underline{\qquad}} = -6$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2 = 75$

Use the Square Root Property.	$y-9=\pm\sqrt{75}$
Simplify the radical.	$y-9=\pm 5\sqrt{3}$
Solve for y.	$y=9\pm 5\sqrt{3}$
Check. $y^2-18y=-6$; $(9+5\sqrt{3})^2-18(9+5\sqrt{3})\stackrel{?}{=}-6$; $81+90\sqrt{3}+75-162+90\sqrt{3}\stackrel{?}{=}-6$; $-6=-6$ ✓	$y^2-18y=-6$; $(9-5\sqrt{3})^2-18(9-5\sqrt{3})\stackrel{?}{=}-6$; $81+90\sqrt{3}+75-162+90\sqrt{3}\stackrel{?}{=}-6$; $-6=-6$ ✓

Another way to check this would be to use a calculator. Evaluate y^2-18y for both of the solutions. The answer should be -6.

TRY IT : : 9.27 Solve by completing the square: $x^2-16x=-16$.

TRY IT : : 9.28 Solve by completing the square: $y^2+8y=11$.

We will start the next example by isolating the variable terms on the left side of the equation.

EXAMPLE 9.15

Solve by completing the square: $x^2+10x+4=15$.

Solution

	$x^2+10x+4=15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2+10x=11$
Take half of 10 and square it.	
$\left(\frac{1}{2}(10)\right)^2=25$	$x^2-10x+\underset{\left(\frac{1}{2}\cdot(10)\right)^2}{\underline{\qquad}}=11$
Add 25 to both sides.	$x^2+10x+25=11+25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2=36$
Use the Square Root Property.	$x+5=\pm\sqrt{36}$
Simplify the radical.	$x+5=\pm 6$
Solve for x.	$x=-5\pm 6$
Rewrite to show two solutions.	$x=-5+6,\quad x=-5-6$
Solve the equations.	$x=1,\quad x=-11$

Check:

$x^2 + 10x + 4 = 15$	$x^2 + 10x + 4 = 15$
$(1)^2 + 10(1) + 4 \stackrel{?}{=} 15$	$(-11)^2 + 10(-11) + 4 \stackrel{?}{=} 15$
$1 + 10 + 4 \stackrel{?}{=} 15$	$121 + 110 + 4 \stackrel{?}{=} 15$
$15 = 15$ ✓	$15 = 15$ ✓

TRY IT : : 9.29 Solve by completing the square: $a^2 + 4a + 9 = 30$.

TRY IT : : 9.30 Solve by completing the square: $b^2 + 8b - 4 = 16$.

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

EXAMPLE 9.16

Solve by completing the square: $n^2 = 3n + 11$.

Solution

	$n^2 = 3n + 11$
Subtract $3n$ to get the variable terms on the left side.	$n^2 - 3n = 11$
Take half of -3 and square it.	
$\left(\frac{1}{2}(-3)\right)^2 = \frac{9}{4}$	$n^2 - 3n + \underline{\left(\frac{1}{2}\cdot(-3)\right)^2} = 11$
Add $\frac{9}{4}$ to both sides.	$n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$
Add the fractions on the right side.	$\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$
Use the Square Root Property.	$n - \frac{3}{2} = \pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n - \frac{3}{2} = \pm\frac{\sqrt{53}}{2}$
Solve for n.	$n = \frac{3}{2} \pm \frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	$n = \frac{3}{2} + \frac{\sqrt{53}}{2}, \quad n = \frac{3}{2} - \frac{\sqrt{53}}{2}$
Check: We leave the check for you!	

TRY IT : : 9.31 Solve by completing the square: $p^2 = 5p + 9$.

TRY IT : : 9.32 Solve by completing the square: $q^2 = 7q - 3$.

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the Zero Product Property since it says "If $a \cdot b = 0$, then *a* = 0 or *b* = 0." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

EXAMPLE 9.17

Solve by completing the square: $(x - 3)(x + 5) = 9$.

Solution

	$(x-3)(x+5)=9$
We multiply the binomials on the left.	$x^2+2x-15=9$
Add 15 to isolate the constant terms on the right.	$x^2+2x \quad =24$
Take half of 2 and square it.	
$\left(\frac{1}{2}\cdot(2)\right)^2=1$	$x^2+2x+\underset{\left(\frac{1}{2}\cdot(2)\right)^2}{\underline{\qquad}}=24$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for *x*.	$x=-1\pm5$
Rewrite to show two solutions.	$x=-1+5,\ x=-1-5$
Simplify.	$x=4, \quad x=-6$
Check: We leave the check for you!	

TRY IT : : 9.33 Solve by completing the square: $(c - 2)(c + 8) = 11$.

TRY IT : : 9.34 Solve by completing the square: $(d - 7)(d + 3) = 56$.

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

The process of completing the square works best when the coefficient of x^2 is 1, so the left side of the equation is of the form $x^2 + bx + c$. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

EXAMPLE 9.18

Solve by completing the square: $3x^2 - 12x - 15 = 0$.

Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2 - 12x - 15 = 0$
Factor out the greatest common factor.	$3(x^2 - 4x - 5) = 0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2 - 4x - 5)}{3} = \frac{0}{3}$
Simplify.	$x^2 - 4x - 5 = 0$
Add 5 to get the constant terms on the right side.	$x^2 - 4x \quad = 5$
Take half of 4 and square it.	
$\left(\frac{1}{2}(-4)\right)^2 = 4$	$x^2 - 4x + \underset{\left(\frac{1}{2}\cdot(4)\right)^2}{\underline{\qquad}} = 5$
Add 4 to both sides.	$x^2 - 4x + 4 = 5 + 4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x - 2)^2 = 9$
Use the Square Root Property.	$x - 2 = \pm\sqrt{9}$
Solve for *x*.	$x - 2 = \pm 3$
Rewrite to show two solutions.	$x = 2 + 3, \; x = 2 - 3$
Simplify.	$x = 5, \quad x = -1$
Check:	

$$\begin{array}{rr} x = 5 & x = -1 \\ 3x^2 - 12x - 15 = 0 & 3x^2 - 12x - 15 = 0 \\ 3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0 & 3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0 \\ 75 - 60 - 15 \stackrel{?}{=} 0 & 3 + 12 - 15 \stackrel{?}{=} 0 \\ 0 = 0 \checkmark & 0 = 0 \checkmark \end{array}$$

TRY IT : : 9.35 Solve by completing the square: $2m^2 + 16m + 14 = 0$.

TRY IT : : 9.36 Solve by completing the square: $4n^2 - 24n - 56 = 8$.

To complete the square, the coefficient of the x^2 must be 1. When the leading coefficient is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

EXAMPLE 9.19

Solve by completing the square: $2x^2 - 3x = 20$.

Solution

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	
$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$	$x^2 - \frac{3}{2}x + \underline{\quad\quad} = 10$ $\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for *x*.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4}, \quad x = \frac{3}{4} - \frac{13}{4}$
Simplify.	$x = 4, \quad x = -\frac{5}{2}$
Check: We leave the check for you!	

TRY IT : : 9.37 Solve by completing the square: $3r^2 - 2r = 21$.

TRY IT : : 9.38 Solve by completing the square: $4t^2 + 2t = 20$.

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a quadratic equation by completing the square to include equations of the form $ax^2 + bx + c = 0$.

HOW TO : : SOLVE A QUADRATIC EQUATION OF THE FORM $ax^2 + bx + c = 0$ BY COMPLETING THE SQUARE.

Step 1. Divide by a to make the coefficient of x^2 term 1.

Step 2. Isolate the variable terms on one side and the constant terms on the other.

Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right

Step 5. Use the Square Root Property.

Step 6. Simplify the radical and then solve the two resulting equations.

Step 7. Check the solutions.

EXAMPLE 9.20

Solve by completing the square: $3x^2 + 2x = 4$.

Solution

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2 + 2x = 4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2 + 2x}{3} = \frac{4}{3}$
Simplify.	$x^2 + \frac{2}{3}x = \frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2 = \frac{1}{9}$	$x^2 + \frac{2}{3}x + \underline{\quad} = \frac{4}{3}$, $\left(\frac{1}{2} \cdot \frac{2}{3}\right)^2$
Add $\frac{1}{9}$ to both sides.	$x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{4}{3} + \frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x + \frac{1}{3}\right)^2 = \frac{12}{9} + \frac{1}{9}$
Use the Square Root Property.	$x + \frac{1}{3} = \pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x + \frac{1}{3} = \pm\frac{\sqrt{13}}{3}$
Solve for x .	$x = -\frac{1}{3} \pm \frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x = -\frac{1}{3} + \frac{\sqrt{13}}{3}$, $x = -\frac{1}{3} - \frac{\sqrt{13}}{3}$
Check: We leave the check for you!	

TRY IT : : 9.39 Solve by completing the square: $4x^2 + 3x = 2$.

TRY IT : : 9.40 Solve by completing the square: $3y^2 - 10y = -5$.

MEDIA : :

Access these online resources for additional instruction and practice with completing the square.

- **Completing Perfect Square Trinomials (https://openstax.org/l/37CompTheSq1)**
- **Completing the Square 1 (https://openstax.org/l/37CompTheSq2)**
- **Completing the Square to Solve Quadratic Equations (https://openstax.org/l/37CompTheSq3)**
- **Completing the Square to Solve Quadratic Equations: More Examples (https://openstax.org/l/37CompTheSq4)**
- **Completing the Square 4 (https://openstax.org/l/37CompTheSq5)**

9.2 EXERCISES

Practice Makes Perfect

Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

71.
ⓐ $m^2 - 24m$
ⓑ $x^2 - 11x$
ⓒ $p^2 - \frac{1}{3}p$

72.
ⓐ $n^2 - 16n$
ⓑ $y^2 + 15y$
ⓒ $q^2 + \frac{3}{4}q$

73.
ⓐ $p^2 - 22p$
ⓑ $y^2 + 5y$
ⓒ $m^2 + \frac{2}{5}m$

74.
ⓐ $q^2 - 6q$
ⓑ $x^2 - 7x$
ⓒ $n^2 - \frac{2}{3}n$

Solve Quadratic Equations of the form $x^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

75. 5. $u^2 + 2u = 3$

76. $z^2 + 12z = -11$

77. $x^2 - 20x = 21$

78. $y^2 - 2y = 8$

79. $m^2 + 4m = -44$

80. $n^2 - 2n = -3$

81. $r^2 + 6r = -11$

82. $t^2 - 14t = -50$

83. $a^2 - 10a = -5$

84. $b^2 + 6b = 41$

85. $x^2 + 5x = 2$

86. $y^2 - 3y = 2$

87. $u^2 - 14u + 12 = 1$

88. $z^2 + 2z - 5 = 2$

89. $r^2 - 4r - 3 = 9$

90. $t^2 - 10t - 6 = 5$

91. $v^2 = 9v + 2$

92. $w^2 = 5w - 1$

93. $x^2 - 5 = 10x$

94. $y^2 - 14 = 6y$

95. $(x + 6)(x - 2) = 9$

96. $(y + 9)(y + 7) = 80$

97. $(x + 2)(x + 4) = 3$

98. $(x - 2)(x - 6) = 5$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

99. $3m^2 + 30m - 27 = 6$

100. $2x^2 - 14x + 12 = 0$

101. $2n^2 + 4n = 26$

102. $5x^2 + 20x = 15$

103. $2c^2 + c = 6$

104. $3d^2 - 4d = 15$

105. $2x^2 + 7x - 15 = 0$

106. $3x^2 - 14x + 8 = 0$

107. $2p^2 + 7p = 14$

108. $3q^2 - 5q = 9$

109. $5x^2 - 3x = -10$

110. $7x^2 + 4x = -3$

Writing Exercises

111. Solve the equation $x^2 + 10x = -25$

ⓐ by using the Square Root Property

ⓑ by Completing the Square

ⓒ Which method do you prefer? Why?

112. Solve the equation $y^2 + 8y = 48$ by completing the square and explain all your steps.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2 + bx + c = 0$ by completing the square.			
solve quadratic equations of the form $ax^2 + bx + c = 0$ by completing the square.			

ⓑ *After reviewing this checklist, what will you do to become confident for all objectives?*

9.3 Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Be Prepared!

Before you get started, take this readiness quiz.

1. Evaluate $b^2 - 4ab$ when $a = 3$ and $b = -2$.
   If you missed this problem, review **Example 1.21**.
2. Simplify: $\sqrt{108}$.
   If you missed this problem, review **Example 8.13**.
3. Simplify: $\sqrt{50}$.
   If you missed this problem, review **Example 8.76**.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for *x*.

We start with the standard form of a quadratic equation and solve it for *x* by completing the square.

	$ax^2 + bx + c = 0 \qquad a \neq 0$
Isolate the variable terms on one side.	$ax^2 + bx = -c$
Make the coefficient of x^2 equal to 1, by dividing by *a*.	$\frac{ax^2}{a} + \frac{b}{a}x = -\frac{c}{a}$
Simplify.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + = -\frac{c}{a} + \frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot 4a}{a \cdot 4a}$

Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
	This equation is the Quadratic Formula.

Quadratic Formula

The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the Quadratic Formula, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the formula is an equation. Make sure you use both sides of the equation.

EXAMPLE 9.21 HOW TO SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution

Step 1. Write the quadratic equation in standard form. Identify the *a*, *b*, *c* values.	This equation is in standard form.	$ax^2+bx+c=0$ $2x^2+9x-5=0$ $a=2, b=9, c=-5$
Step 2. Write the quadratic formula. Then substitute in the values of *a*, *b*, *c*.	Substitute in $a=2$, $b=9$, $c=-5$	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$

Step 3. Simplify the fraction, and solve for x.		$x = \frac{-9 \pm \sqrt{81-(-40)}}{4}$ $x = \frac{-9 \pm \sqrt{121}}{4}$ $x = \frac{-9 \pm 11}{4}$ $x = \frac{-9+11}{4} \quad x = \frac{-9-11}{4}$ $x = \frac{2}{4} \quad x = \frac{-20}{4}$ $x = \frac{1}{2} \quad x = -5$
Step 4. Check the solutions.	Put each answer in the original equation to check. Substitute $x = \frac{1}{2}$. Substitute $x = -5$.	$2x^2 + 9x - 5 = 0$ $2\left(\frac{1}{2}\right)^2 + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $2 \cdot \frac{1}{4} + 9 \cdot \frac{1}{2} - 5 \stackrel{?}{=} 0$ $\frac{1}{2} + \frac{9}{2} - 5 \stackrel{?}{=} 0$ $\frac{10}{2} - 5 \stackrel{?}{=} 0$ $5 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $2x^2 + 9x - 5 = 0$ $2(-5)^2 + 9(-5) - 5 \stackrel{?}{=} 0$ $2 \cdot 25 - 45 - 5 \stackrel{?}{=} 0$ $50 - 45 - 5 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

TRY IT : : 9.41 Solve by using the Quadratic Formula: $3y^2 - 5y + 2 = 0$.

TRY IT : : 9.42 Solve by using the Quadratic Formula: $4z^2 + 2z - 6 = 0$.

HOW TO : : SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA.

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, and c.

Step 2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.

Step 3. Simplify.

Step 4. Check the solutions.

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic

Formula is an EQUATION. Be sure you start with “$x =$”.

EXAMPLE 9.22

Solve by using the Quadratic Formula: $x^2 - 6x = -5$.

Solution

	$x^2 - 6x = -5$
Write the equation in standard form by adding 5 to each side.	$x^2 - 6x + 5 = 0$
This equation is now in standard form.	$ax^2 + bx + c = 0$ $x^2 - 6x + 5 = 0$
Identify the values of a, b, c.	$a = 1, b = -6, c = 5$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot (5)}}{2 \cdot 1}$
Simplify.	$x = \frac{6 \pm \sqrt{36 - 20}}{2}$ $x = \frac{6 \pm \sqrt{16}}{2}$ $x = \frac{6 \pm 4}{2}$
Rewrite to show two solutions.	$x = \frac{6 + 4}{2}, \quad x = \frac{6 - 4}{2}$
Simplify.	$x = \frac{10}{2}, \quad x = \frac{2}{2}$
	$x = 5, \quad x = 1$

Check:

$x^2 - 6x + 5 = 0$
$5^2 - 6 \cdot 5 + 5 \stackrel{?}{=} 0$
$25 - 30 + 5 \stackrel{?}{=} 0$
$0 = 0 ✓$

$x^2 - 6x + 5 = 0$
$1^2 - 6 \cdot 1 + 5 \stackrel{?}{=} 0$
$1 - 6 + 5 \stackrel{?}{=} 0$
$0 = 0 ✓$

TRY IT : : 9.43 Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

TRY IT : : 9.44 Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the Quadratic Formula. If we get a radical as a solution, the final answer must have the radical in its simplified form.

EXAMPLE 9.23

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$\begin{matrix} ax^2 + bx + c = 0 \\ 2x^2 + 10x + 11 = 0 \end{matrix}$
Identify the values of *a*, *b*, and *c*.	$a = 2,\ b = 10,\ c = 11$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, and *c*.	$x = \dfrac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot (11)}}{2 \cdot 2}$
Simplify.	$x = \dfrac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \dfrac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \dfrac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \dfrac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \dfrac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \dfrac{-5 + \sqrt{3}}{2},\quad x = \dfrac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

TRY IT : : 9.45 Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

TRY IT : : 9.46 Solve by using the Quadratic Formula: $5n^2 + 4n - 4 = 0$.

When we substitute *a*, *b*, and *c* into the Quadratic Formula and the radicand is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

EXAMPLE 9.24

Solve by using the Quadratic Formula: $3p^2 + 2p + 9 = 0.$

Solution

	$3p^2 + 2p + 9 = 0$
This equation is in standard form	$\begin{matrix} ax^2 + bx + c = 0 \\ 3p^2 + 2p + 9 = 0 \end{matrix}$
Identify the values of $a,\ b,\ c$.	$a = 3,\ b = 2,\ c = 9$
Write the Quadratic Formula.	$p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of $a,\ b,\ c$.	$p = \dfrac{-(2) \pm \sqrt{(2)^2 - 4 \cdot 3 \cdot (9)}}{2 \cdot 3}$

Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$
	$p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}\,i}{6}$
Simplify the radical.	$p=\frac{-2\pm 2\sqrt{26}\,i}{6}$
Factor the common factor in the numerator.	$p=\frac{2\left(-1\pm\sqrt{26}\,i\right)}{6}$
Remove the common factors.	$p=\frac{-1\pm\sqrt{26}\,i}{3}$
Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

TRY IT : : 9.47 Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

TRY IT : : 9.48 Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

EXAMPLE 9.25

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution

Our first step is to get the equation in standard form.

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$

Factor the common factor in the numerator.	$x = \frac{2\left(-3 \pm 2\sqrt{5}\right)}{2}$
Remove the common factors.	$x = -3 \pm 2\sqrt{5}$
Write as two solutions.	$x = -3 + 2\sqrt{5},\ \ x = -3 - 2\sqrt{5}$
Check: We leave the check for you!	

TRY IT : : 9.49 Solve by using the Quadratic Formula: $x(x+2) - 5 = 0.$

TRY IT : : 9.50 Solve by using the Quadratic Formula: $3y(y-2) - 3 = 0.$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

EXAMPLE 9.26

Solve by using the Quadratic Formula: $\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}.$

Solution

Our first step is to clear the fractions.

	$\frac{1}{2}u^2 + \frac{2}{3}u = \frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2 + \frac{2}{3}u\right) = 6\left(\frac{1}{3}\right)$
Multiply.	$3u^2 + 4u = 2$
Subtract 2 to get the equation in standard form.	$ax^2 + bx + c = 0$ $3u^2 + 4u - 2 = 0$
Identify the values of *a*, *b*, and *c*.	$a = 3, b = 4, c = -2$
Write the Quadratic Formula.	$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, and *c*.	$u = \frac{-(4) \pm \sqrt{(4)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3}$
Simplify.	$u = \frac{-4 \pm \sqrt{16 + 24}}{6}$
	$u = \frac{-4 \pm \sqrt{40}}{6}$
Simplify the radical.	$u = \frac{-4 \pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u = \frac{2\left(-2 \pm \sqrt{10}\right)}{6}$
Remove the common factors.	$u = \frac{-2 \pm \sqrt{10}}{3}$
Rewrite to show two solutions.	$u = \frac{-2 + \sqrt{10}}{3}, \quad u = \frac{-2 - \sqrt{10}}{3}$

Check:
We leave the check for you!

TRY IT : : 9.51 Solve by using the Quadratic Formula: $\frac{1}{4}c^2 - \frac{1}{3}c = \frac{1}{12}$.

TRY IT : : 9.52 Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Think about the equation $(x - 3)^2 = 0$. We know from the Zero Product Property that this equation has only one solution, $x = 3$.

We will see in the next example how using the Quadratic Formula to solve an equation whose standard form is a perfect square trinomial equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0 , which leads to only one solution.

EXAMPLE 9.27

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$\begin{matrix} ax^2 + bx + c = 0 \\ 4x^2 - 20x + 25 = 0 \end{matrix}$
Identify the values of *a*, *b*, and *c*.	$a = 4,\ b = -20,\ c = 25$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of *a*, *b*, and *c*.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4 \cdot 4 \cdot (25)}}{2 \cdot 4}$
Simplify.	$x = \frac{20 \pm \sqrt{400 - 400}}{8}$
	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the radical.	$x = \frac{20}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x - 5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

TRY IT : : 9.53 Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

TRY IT : : 9.54 Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Discriminant

In the Quadratic Formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the quantity $b^2 - 4ac$ is called the discriminant.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Value of the Discriminant	Number and Type of solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	+	2 real
$4x^2 - 20x + 25 = 0$	$(-20)^2 - 4 \cdot 4 \cdot 25$ 0	0	1 real
$3p^2 + 2p + 9 = 0$	$2^2 - 4 \cdot 3 \cdot 9$ -104	–	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**. $x = \dfrac{-b \pm \sqrt{+}}{2a}$

When the discriminant is zero, the quadratic equation has **1 real solution**. $x = \dfrac{-b \pm \sqrt{0}}{2a}$

When the discriminant is negative, the quadratic equation has **2 complex solutions**. $x = \dfrac{-b \pm \sqrt{-}}{2a}$

Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

EXAMPLE 9.28

Determine the number of solutions to each quadratic equation.

ⓐ $3x^2 + 7x - 9 = 0$ ⓑ $5n^2 + n + 4 = 0$ ⓒ $9y^2 - 6y + 1 = 0$.

Solution

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

ⓐ

	$3x^2 + 7x - 9 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 3, \quad b = 7, \quad c = -9$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of $a, b,$ and c.	$(7)^2 - 4 \cdot 3 \cdot (-9)$
Simplify.	$49 + 108$
	157

Since the discriminant is positive, there are 2 real solutions to the equation.

ⓑ

	$5n^2 + n + 4 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 5, \quad b = 1, \quad c = 4$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of $a, b,$ and c.	$(1)^2 - 4 \cdot 5 \cdot 4$
Simplify.	$1 - 80$
	-79

Since the discriminant is negative, there are 2 complex solutions to the equation.

ⓒ

	$9y^2 - 6y + 1 = 0$
The equation is in standard form, identify $a, b,$ and c.	$a = 9, \; b = -6, \; c = 1$
Write the discriminant.	$b^2 - 4ac$
Substitute in the values of $a, b,$ and c.	$(-6)^2 - 4 \cdot 9 \cdot 1$
Simplify.	$36 - 36$
	0

Since the discriminant is 0, there is 1 real solution to the equation.

TRY IT : : 9.55 Determine the numberand type of solutions to each quadratic equation.

ⓐ $8m^2 - 3m + 6 = 0$ ⓑ $5z^2 + 6z - 2 = 0$ ⓒ $9w^2 + 24w + 16 = 0.$

TRY IT : : 9.56 Determine the number and type of solutions to each quadratic equation.

ⓐ $b^2 + 7b - 13 = 0$ ⓑ $5a^2 - 6a + 10 = 0$ ⓒ $4r^2 - 20r + 25 = 0.$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2 = k$ or $a(x - h)^2 = k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

HOW TO : : IDENTIFY THE MOST APPROPRIATE METHOD TO SOLVE A QUADRATIC EQUATION.

Step 1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.

Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.

Step 3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

EXAMPLE 9.29

Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $5z^2 = 17$ ⓑ $4x^2 - 12x + 9 = 0$ ⓒ $8u^2 + 6u = 11$.

Solution

ⓐ

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

ⓑ

$$4x^2 - 12x + 9 = 0$$

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

ⓒ

	$8u^2 + 6u = 11$
Put the equation in standard form.	$8u^2 + 6u - 11 = 0$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

TRY IT : : 9.57 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $x^2 + 6x + 8 = 0$ ⓑ $(n-3)^2 = 16$ ⓒ $5p^2 - 6p = 9$.

TRY IT : : 9.58 Identify the most appropriate method to use to solve each quadratic equation.

ⓐ $8a^2 + 3a - 9 = 0$ ⓑ $4b^2 + 4b + 1 = 0$ ⓒ $5c^2 = 125$.

MEDIA : :

Access these online resources for additional instruction and practice with using the Quadratic Formula.

- **Using the Quadratic Formula (https://openstax.org/l/37QuadForm1)**
- **Solve a Quadratic Equation Using the Quadratic Formula with Complex Solutions (https://openstax.org/l/37QuadForm2)**
- **Discriminant in Quadratic Formula (https://openstax.org/l/37QuadForm3)**

9.3 EXERCISES

Practice Makes Perfect

Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

113. $4m^2 + m - 3 = 0$

114. $4n^2 - 9n + 5 = 0$

115. $2p^2 - 7p + 3 = 0$

116. $3q^2 + 8q - 3 = 0$

117. $p^2 + 7p + 12 = 0$

118. $q^2 + 3q - 18 = 0$

119. $r^2 - 8r = 33$

120. $t^2 + 13t = -40$

121. $3u^2 + 7u - 2 = 0$

122. $2p^2 + 8p + 5 = 0$

123. $2a^2 - 6a + 3 = 0$

124. $5b^2 + 2b - 4 = 0$

125. $x^2 + 8x - 4 = 0$

126. $y^2 + 4y - 4 = 0$

127. $3y^2 + 5y - 2 = 0$

128. $6x^2 + 2x - 20 = 0$

129. $2x^2 + 3x + 3 = 0$

130. $2x^2 - x + 1 = 0$

131. $8x^2 - 6x + 2 = 0$

132. $8x^2 - 4x + 1 = 0$

133. $(v + 1)(v - 5) - 4 = 0$

134. $(x + 1)(x - 3) = 2$

135. $(y + 4)(y - 7) = 18$

136. $(x + 2)(x + 6) = 21$

137. $\frac{1}{3}m^2 + \frac{1}{12}m = \frac{1}{4}$

138. $\frac{1}{3}n^2 + n = -\frac{1}{2}$

139. $\frac{3}{4}b^2 + \frac{1}{2}b = \frac{3}{8}$

140. $\frac{1}{9}c^2 + \frac{2}{3}c = 3$

141. $16c^2 + 24c + 9 = 0$

142. $25d^2 - 60d + 36 = 0$

143. $25q^2 + 30q + 9 = 0$

144. $16y^2 + 8y + 1 = 0$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

145.
ⓐ $4x^2 - 5x + 16 = 0$
ⓑ $36y^2 + 36y + 9 = 0$
ⓒ $6m^2 + 3m - 5 = 0$

146.
ⓐ $9v^2 - 15v + 25 = 0$
ⓑ $100w^2 + 60w + 9 = 0$
ⓒ $5c^2 + 7c - 10 = 0$

147.
ⓐ $r^2 + 12r + 36 = 0$
ⓑ $8t^2 - 11t + 5 = 0$
ⓒ $3v^2 - 5v - 1 = 0$

148.
ⓐ $25p^2 + 10p + 1 = 0$
ⓑ $7q^2 - 3q - 6 = 0$
ⓒ $7y^2 + 2y + 8 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

149.
ⓐ $x^2 - 5x - 24 = 0$
ⓑ $(y + 5)^2 = 12$
ⓒ $14m^2 + 3m = 11$

150.
ⓐ $(8v + 3)^2 = 81$
ⓑ $w^2 - 9w - 22 = 0$
ⓒ $4n^2 - 10 = 6$

151.
ⓐ $6a^2 + 14 = 20$
ⓑ $\left(x - \frac{1}{4}\right)^2 = \frac{5}{16}$
ⓒ $y^2 - 2y = 8$

152.
ⓐ $8b^2 + 15b = 4$
ⓑ $\frac{5}{9}v^2 - \frac{2}{3}v = 1$
ⓒ $\left(w + \frac{4}{3}\right)^2 = \frac{2}{9}$

Writing Exercises

153. Solve the equation $x^2 + 10x = 120$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

154. Solve the equation $12y^2 + 23y = 24$
ⓐ by completing the square
ⓑ using the Quadratic Formula
ⓒ Which method do you prefer? Why?

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

ⓑ *What does this checklist tell you about your mastery of this section? What steps will you take to improve?*

9.4 Solve Quadratic Equations in Quadratic Form

Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

Be Prepared!

Before you get started, take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20.$
   If you missed this problem, review Example 6.21.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15.$
   If you missed this problem, review Example 6.22.
3. Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$ ⓑ $\left(x^{\frac{1}{3}}\right)^2$ ⓒ $\left(x^{-1}\right)^2.$

   If you missed this problem, review Example 8.33.

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$
	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$(x^2+1)(x^2-5)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

EXAMPLE 9.30 HOW TO SOLVE EQUATIONS IN QUADRATIC FORM

Solve: $6x^4 - 7x^2 + 2 = 0$

Solution

Step 1. Identify a substitution that will put the equation in quadratic form.	Since $(x^2)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$
Step 2. Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6(x^2)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3. Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u - 1)(3u - 2) = 0$ $2u - 1 = 0,\ 3u - 2 = 0$ $2u = 1,\ 3u = 2$ $u = \frac{1}{2} \quad u = \frac{2}{3}$
Step 4. Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \qquad x^2 = \frac{2}{3}$
Step 5. Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}} \qquad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2} \qquad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2} \qquad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2} \qquad x = -\frac{\sqrt{6}}{3}$
Step 6. Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0\ ✓$ We leave the other checks to you!

TRY IT : : 9.59 Solve: $x^4 - 6x^2 + 8 = 0$.

TRY IT : : 9.60 Solve: $x^4 - 11x^2 + 28 = 0$.

We summarize the steps to solve an equation in quadratic form.

HOW TO : : SOLVE EQUATIONS IN QUADRATIC FORM.

Step 1. Identify a substitution that will put the equation in quadratic form.
Step 2. Rewrite the equation with the substitution to put it in quadratic form.
Step 3. Solve the quadratic equation for u.
Step 4. Substitute the original variable back into the results, using the substitution.
Step 5. Solve for the original variable.
Step 6. Check the solutions.

In the next example, the binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

EXAMPLE 9.31

Solve: $(x-2)^2+7(x-2)+12=0.$

Solution

	$(x-2)^2+7(x-2)+12=0$
Prepare for the substitution.	$(x-2)^2+7(x-2)+12=0$
Let $u=x-2$ and substitute.	$u^2+7u+12=0$
Solve by factoring.	$(u+3)(u+4)=0$ $u+3=0,\quad u+4=0$ $u=-3,\qquad u=-4$
Replace u with $x-2$.	$x-2=-3,\quad x-2=-4$
Solve for x.	$x=-1,\qquad x=-2$

Check:

$x=-1$	$x=-2$
$(x-2)^2+7(x-2)+12=0$	$(x-2)^2+7(x-2)+12=0$
$(-1-2)^2+7(-1-2)+12\stackrel{?}{=}0$	$(-2-2)^2+7(-2-2)+12\stackrel{?}{=}0$
$(-3)^2+7(-3)+12\stackrel{?}{=}0$	$(-4)^2+7(-4)+12\stackrel{?}{=}0$
$9-21+12\stackrel{?}{=}0$	$16-28+12\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

TRY IT : : 9.61 Solve: $(x-5)^2+6(x-5)+8=0.$

TRY IT : : 9.62 Solve: $(y-4)^2+8(y-4)+15=0.$

In the next example, we notice that $(\sqrt{x})^2=x.$ Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

EXAMPLE 9.32

Solve: $x - 3\sqrt{x} + 2 = 0$.

Solution

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2 = x$. If we let $u = \sqrt{x}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x - 3\sqrt{x} + 2 = 0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2 - 3\sqrt{x} + 2 = 0$
Let $u = \sqrt{x}$ and substitute.	$u^2 - 3u + 2 = 0$
Solve by factoring.	$(u-2)(u-1) = 0$ $u - 2 = 0, \quad u - 1 = 0$ $u = 2, \quad u = 1$
Replace u with $\sqrt{x}$.	$\sqrt{x} = 2, \quad \sqrt{x} = 1$
Solve for x, by squaring both sides.	$x = 4, \quad x = 1$

Check:

$x = 4$	$x = 1$
$x - 3\sqrt{x} + 2 = 0$	$x - 3\sqrt{x} + 2 = 0$
$4 - 3\sqrt{4} + 2 \stackrel{?}{=} 0$	$1 - 3\sqrt{1} + 2 \stackrel{?}{=} 0$
$4 - 6 + 2 \stackrel{?}{=} 0$	$1 - 3 + 2 \stackrel{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

TRY IT : : 9.63 Solve: $x - 7\sqrt{x} + 12 = 0$.

TRY IT : : 9.64 Solve: $x - 6\sqrt{x} + 8 = 0$.

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

EXAMPLE 9.33

Solve: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$.

Solution

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$. If we let $u = x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$
Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$
Let $u = x^{\frac{1}{3}}$ and substitute.	$u^2 - 2u - 24 = 0$

Solve by factoring.	$(u-6)(u+4)=0$
	$u-6=0,\quad u+4=0$
	$u=6,\quad u=-4$
Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}}=6,\quad x^{\frac{1}{3}}=-4$
Solve for x by cubing both sides.	$\left(x^{\frac{1}{3}}\right)^3=(6)^3,\quad \left(x^{\frac{1}{3}}\right)^3=(-4)^3$
	$x=216,\quad x=-64$

Check:

$x=216$	$x=-64$
$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$	$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$
$(216)^{\frac{2}{3}}-2(216)^{\frac{1}{3}}-24\stackrel{?}{=}0$	$(-64)^{\frac{2}{3}}-2(-64)^{\frac{1}{3}}-24\stackrel{?}{=}0$
$36-12-24\stackrel{?}{=}0$	$16+8-24\stackrel{?}{=}0$
$0=0\checkmark$	$0=0\checkmark$

TRY IT : : 9.65 Solve: $x^{\frac{2}{3}}-5x^{\frac{1}{3}}-14=0.$

TRY IT : : 9.66 Solve: $x^{\frac{1}{2}}+8x^{\frac{1}{4}}+15=0.$

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

EXAMPLE 9.34

Solve: $3x^{-2}-7x^{-1}+2=0.$

Solution

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2=x^{-2}$. If we let $u=x^{-1}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$3x^{-2}-7x^{-1}+2=0$
Rewrite the trinomial to prepare for the substitution.	$3(x^{-1})^2-7(x^{-1})+2=0$
Let $u=x^{-1}$ and substitute.	$3u^2-7u+2=0$
Solve by factoring.	$(3u-1)(u-2)=0$
	$3u-1=0,\quad u-2=0$
	$u=\frac{1}{3},\quad u=2$
Replace u with x^{-1}.	$x^{-1}=\frac{1}{3},\quad x^{-1}=2$
Solve for x by taking the reciprocal since $x^{-1}=\frac{1}{x}$.	$x=3,\quad x=\frac{1}{2}$

Check:

$$x = 3 \qquad\qquad x = \frac{1}{2}$$

$$3x^{-2} - 7x^{-1} + 2 = 0 \qquad\qquad 3x^{-2} - 7x^{-1} + 2 = 0$$

$$3(3)^{-2} - 7(3)^{-1} + 2 \stackrel{?}{=} 0 \qquad\qquad 3\left(\frac{1}{2}\right)^{-2} - 7\left(\frac{1}{2}\right)^{-1} + 2 \stackrel{?}{=} 0$$

$$3\left(\frac{1}{9}\right) - 7\left(\frac{1}{3}\right) + 2 \stackrel{?}{=} 0 \qquad\qquad 3(4) - 7(2) + 2 \stackrel{?}{=} 0$$

$$\frac{1}{3} - \left(\frac{7}{3}\right) + \frac{6}{3} \stackrel{?}{=} 0 \qquad\qquad 12 - 14 + 2 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark \qquad\qquad 0 = 0 \checkmark$$

TRY IT : : 9.67 Solve: $8x^{-2} - 10x^{-1} + 3 = 0.$

TRY IT : : 9.68 Solve: $6x^{-2} - 23x^{-1} + 20 = 0.$

MEDIA : :

Access this online resource for additional instruction and practice with solving quadratic equations.

- **Solving Equations in Quadratic Form (https://openstax.org/l/37QuadForm4)**

9.4 EXERCISES

Practice Makes Perfect

Solve Equations in Quadratic Form

In the following exercises, solve.

155. $x^4 - 7x^2 + 12 = 0$

156. $x^4 - 9x^2 + 18 = 0$

157. $x^4 - 13x^2 - 30 = 0$

158. $x^4 + 5x^2 - 36 = 0$

159. $2x^4 - 5x^2 + 3 = 0$

160. $4x^4 - 5x^2 + 1 = 0$

161. $2x^4 - 7x^2 + 3 = 0$

162. $3x^4 - 14x^2 + 8 = 0$

163. $(x-3)^2 - 5(x-3) - 36 = 0$

164. $(x+2)^2 - 3(x+2) - 54 = 0$

165. $(3y+2)^2 + (3y+2) - 6 = 0$

166. $(5y-1)^2 + 3(5y-1) - 28 = 0$

167. $\left(x^2+1\right)^2 - 5\left(x^2+1\right) + 4 = 0$

168. $\left(x^2-4\right)^2 - 4\left(x^2-4\right) + 3 = 0$

169. $2\left(x^2-5\right)^2 - 5\left(x^2-5\right) + 2 = 0$

170. $2\left(x^2-5\right)^2 - 7\left(x^2-5\right) + 6 = 0$

171. $x - \sqrt{x} - 20 = 0$

172. $x - 8\sqrt{x} + 15 = 0$

173. $x + 6\sqrt{x} - 16 = 0$

174. $x + 4\sqrt{x} - 21 = 0$

175. $6x + \sqrt{x} - 2 = 0$

176. $6x + \sqrt{x} - 1 = 0$

177. $10x - 17\sqrt{x} + 3 = 0$

178. $12x + 5\sqrt{x} - 3 = 0$

179. $x^{\frac{2}{3}} + 9x^{\frac{1}{3}} + 8 = 0$

180. $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 28$

181. $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} = 12$

182. $x^{\frac{2}{3}} - 11x^{\frac{1}{3}} + 30 = 0$

183. $6x^{\frac{2}{3}} - x^{\frac{1}{3}} = 12$

184. $3x^{\frac{2}{3}} - 10x^{\frac{1}{3}} = 8$

185. $8x^{\frac{2}{3}} - 43x^{\frac{1}{3}} + 15 = 0$

186. $20x^{\frac{2}{3}} - 23x^{\frac{1}{3}} + 6 = 0$

187. $x + 8x^{\frac{1}{2}} + 7 = 0$

188. $2x - 7x^{\frac{1}{2}} = 15$

189. $6x^{-2} + 13x^{-1} + 5 = 0$

190. $15x^{-2} - 26x^{-1} + 8 = 0$

191. $8x^{-2} - 2x^{-1} - 3 = 0$

192. $15x^{-2} - 4x^{-1} - 4 = 0$

Writing Exercises

193. Explain how to recognize an equation in quadratic form.

194. Explain the procedure for solving an equation in quadratic form.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*

CHAPTER 9 REVIEW

KEY TERMS

discriminant In the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the quantity $b^2 - 4ac$ is called the discriminant.

quadratic function A quadratic function, where *a*, *b*, and *c* are real numbers and $a \neq 0$, is a function of the form $f(x) = ax^2 + bx + c$.

quadratic inequality A quadratic inequality is an inequality that contains a quadratic expression.

KEY CONCEPTS

9.1 Solve Quadratic Equations Using the Square Root Property

- Square Root Property
    - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$ or $x = \pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  Step 1. Isolate the quadratic term and make its coefficient one.

  Step 2. Use Square Root Property.

  Step 3. Simplify the radical.

  Step 4. Check the solutions.

9.2 Solve Quadratic Equations by Completing the Square

- Binomial Squares Pattern
  If *a* and *b* are real numbers,

  $(a+b)^2 = a^2 + 2ab + b^2$ — $\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\,\cdot\,(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$

  $(a-b)^2 = a^2 - 2ab + b^2$ — $\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\,\cdot\,(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$

- How to Complete a Square

  Step 1. Identify *b*, the coefficient of *x*.

  Step 2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.

  Step 3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$

  Step 4. Rewrite the trinomial as a binomial square

- How to solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square.

  Step 1. Divide by *a* to make the coefficient of x^2 term 1.

  Step 2. Isolate the variable terms on one side and the constant terms on the other.

  Step 3. Find $\left(\frac{1}{2} \cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.

  Step 4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.

  Step 5. Use the Square Root Property.

  Step 6. Simplify the radical and then solve the two resulting equations.

Step 7. Check the solutions.

9.3 Solve Quadratic Equations Using the Quadratic Formula

- Quadratic Formula
    - The solutions to a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$ are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.

  Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$. Identify the values of a, b, c.

  Step 2. Write the Quadratic Formula. Then substitute in the values of a, b, c.

  Step 3. Simplify.

  Step 4. Check the solutions.
- Using the Discriminant, $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation
    - For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,
        - If $b^2 - 4ac > 0$, the equation has 2 real solutions.
        - if $b^2 - 4ac = 0$, the equation has 1 real solution.
        - if $b^2 - 4ac < 0$, the equation has 2 complex solutions.
- Methods to Solve Quadratic Equations:
    - Factoring
    - Square Root Property
    - Completing the Square
    - Quadratic Formula
- How to identify the most appropriate method to solve a quadratic equation.

  Step 1. Try Factoring first. If the quadratic factors easily, this method is very quick.

  Step 2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.

  Step 3. Use the **Quadratic Formula.** Any other quadratic equation is best solved by using the Quadratic Formula.

9.4 Solve Quadratic Equations in Quadratic Form

- How to solve equations in quadratic form.

  Step 1. Identify a substitution that will put the equation in quadratic form.

  Step 2. Rewrite the equation with the substitution to put it in quadratic form.

  Step 3. Solve the quadratic equation for u.

  Step 4. Substitute the original variable back into the results, using the substitution.

  Step 5. Solve for the original variable.

  Step 6. Check the solutions.

9.5 Solve Applications of Quadratic Equations

- Methods to Solve Quadratic Equations
    - Factoring
    - Square Root Property
    - Completing the Square
    - Quadratic Formula
- How to use a Problem-Solving Strategy.

  Step 1. **Read** the problem. Make sure all the words and ideas are understood.

  Step 2. **Identify** what we are looking for.

  Step 3. **Name** what we are looking for. Choose a variable to represent that quantity.

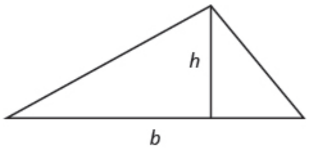
Step 4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.

Step 5. **Solve** the equation using good algebra techniques.

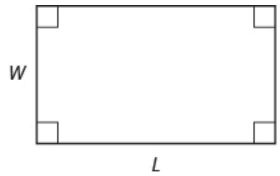
Step 6. **Check** the answer in the problem and make sure it makes sense.

Step 7. **Answer** the question with a complete sentence.

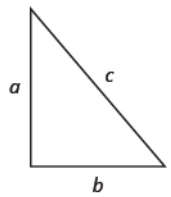
- Area of a Triangle
    - For a triangle with base, *b*, and height, *h*, the area, *A*, is given by the formula $A = \frac{1}{2}bh$.



- Area of a Rectangle
    - For a rectangle with length, *L*, and width, *W*, the area, *A*, is given by the formula $A = LW$.



- Pythagorean Theorem
    - In any right triangle, where *a* and *b* are the lengths of the legs, and *c* is the length of the hypotenuse, $a^2 + b^2 = c^2$.



- Projectile motion
    - The height in feet, *h*, of an object shot upwards into the air with initial velocity, v_0, after *t* seconds is given by the formula $h = -16t^2 + v_0 t$.

9.6 Graph Quadratic Functions Using Properties

- Parabola Orientation
    - For the graph of the quadratic function $f(x) = ax^2 + bx + c$, if
        - $a > 0$, the parabola opens upward.
        - $a < 0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the function $f(x) = ax^2 + bx + c$ is a parabola where:
    - the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
    - the vertex is a point on the axis of symmetry, so its *x*-coordinate is $-\frac{b}{2a}$.
    - the *y*-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
    - To find the intercepts of a parabola whose function is $f(x) = ax^2 + bx + c$:

y-intercept	**x-intercepts**
Let $x = 0$ and solve for $f(x)$.	Let $f(x) = 0$ and solve for x.

- How to graph a quadratic function using properties.
  Step 1. Determine whether the parabola opens upward or downward.
  Step 2. Find the equation of the axis of symmetry.
  Step 3. Find the vertex.
  Step 4. Find the *y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Step 5. Find the *x*-intercepts. Find additional points if needed.
  Step 6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Equation
  - The *y*-coordinate of the vertex of the graph of a quadratic equation is the
  - *minimum* value of the quadratic equation if the parabola opens *upward*.
  - *maximum* value of the quadratic equation if the parabola opens *downward*.

9.7 Graph Quadratic Functions Using Transformations

- Graph a Quadratic Function of the form $f(x) = x^2 + k$ Using a Vertical Shift
  - The graph of $f(x) = x^2 + k$ shifts the graph of $f(x) = x^2$ vertically k units.
    - If $k > 0$, shift the parabola vertically up k units.
    - If $k < 0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic Function of the form $f(x) = (x - h)^2$ Using a Horizontal Shift
  - The graph of $f(x) = (x - h)^2$ shifts the graph of $f(x) = x^2$ horizontally h units.
    - If $h > 0$, shift the parabola horizontally left h units.
    - If $h < 0$, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic Function of the form $f(x) = ax^2$
  - The coefficient *a* in the function $f(x) = ax^2$ affects the graph of $f(x) = x^2$ by stretching or compressing it.
    If $0 < |a| < 1$, then the graph of $f(x) = ax^2$ will be "wider" than the graph of $f(x) = x^2$.
    If $|a| > 1$, then the graph of $f(x) = ax^2$ will be "skinnier" than the graph of $f(x) = x^2$.
- How to graph a quadratic function using transformations
  Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form by completing the square.
  Step 2. Graph the function using transformations.
- Graph a quadratic function in the vertex form $f(x) = a(x - h)^2 + k$ using properties
  Step 1. Rewrite the function in $f(x) = a(x - h)^2 + k$ form.
  Step 2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
  Step 3. Find the axis of symmetry, $x = h$.
  Step 4. Find the vertex, (h, k).
  Step 5. Find the*y*-intercept. Find the point symmetric to the *y*-intercept across the axis of symmetry.
  Step 6. Find the *x*-intercepts, if possible.
  Step 7. Graph the parabola.

9.8 Solve Quadratic Inequalities

- Solve a Quadratic Inequality Graphically
  Step 1. Write the quadratic inequality in standard form.
  Step 2. Graph the function $f(x) = ax^2 + bx + c$ using properties or transformations.
  Step 3. Determine the solution from the graph.
- How to Solve a Quadratic Inequality Algebraically
  Step 1. Write the quadratic inequality in standard form.
  Step 2. Determine the critical points -- the solutions to the related quadratic equation.
  Step 3. Use the critical points to divide the number line into intervals.
  Step 4. Above the number line show the sign of each quadratic expression using test points from each interval substituted into the original inequality.
  Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.

REVIEW EXERCISES

9.1 Solve Quadratic Equations Using the Square Root Property

Solve Quadratic Equations of the form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

395. $y^2 = 144$

396. $n^2 - 80 = 0$

397. $4a^2 = 100$

398. $2b^2 = 72$

399. $r^2 + 32 = 0$

400. $t^2 + 18 = 0$

401. $\frac{2}{3}w^2 - 20 = 30$

402. 11. $5c^2 + 3 = 19$

Solve Quadratic Equations of the Form $a(x - h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

403. $(p - 5)^2 + 3 = 19$

404. $(u + 1)^2 = 45$

405. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$

406. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$

407. $(n - 4)^2 - 50 = 150$

408. $(4c - 1)^2 = -18$

409. $n^2 + 10n + 25 = 12$

410. $64a^2 + 48a + 9 = 81$

9.2 Solve Quadratic Equations by Completing the Square

Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

411. $x^2 + 22x$

412. $m^2 - 8m$

413. $a^2 - 3a$

414. $b^2 + 13b$

In the following exercises, solve by completing the square.

415. $d^2 + 14d = -13$

416. $y^2 - 6y = 36$

417. $m^2 + 6m = -109$

418. $t^2 - 12t = -40$

419. $v^2 - 14v = -31$

420. $w^2 - 20w = 100$

421. $m^2 + 10m - 4 = -13$

422. $n^2 - 6n + 11 = 34$

423. $a^2 = 3a + 8$

424. $b^2 = 11b - 5$

425. $(u + 8)(u + 4) = 14$

426. $(z - 10)(z + 2) = 28$

Solve Quadratic Equations of the form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

427. $3p^2 - 18p + 15 = 15$

428. $5q^2 + 70q + 20 = 0$

429. $4y^2 - 6y = 4$

430. $2x^2 + 2x = 4$

431. $3c^2 + 2c = 9$

432. $4d^2 - 2d = 8$

433. $2x^2 + 6x = -5$

434. $2x^2 + 4x = -5$

9.3 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

435. $4x^2 - 5x + 1 = 0$

436. $7y^2 + 4y - 3 = 0$

437. $r^2 - r - 42 = 0$

438. $t^2 + 13t + 22 = 0$

439. $4v^2 + v - 5 = 0$

440. $2w^2 + 9w + 2 = 0$

441. $3m^2 + 8m + 2 = 0$

442. $5n^2 + 2n - 1 = 0$

443. $6a^2 - 5a + 2 = 0$

444. $4b^2 - b + 8 = 0$

445. $u(u - 10) + 3 = 0$

446. $5z(z - 2) = 3$

447. $\frac{1}{8}p^2 - \frac{1}{5}p = -\frac{1}{20}$

448. $\frac{2}{5}q^2 + \frac{3}{10}q = \frac{1}{10}$

449. $4c^2 + 4c + 1 = 0$

450. $9d^2 - 12d = -4$

Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

451.

ⓐ $9x^2 - 6x + 1 = 0$
ⓑ $3y^2 - 8y + 1 = 0$
ⓒ $7m^2 + 12m + 4 = 0$
ⓓ $5n^2 - n + 1 = 0$

452.

ⓐ $5x^2 - 7x - 8 = 0$
ⓑ $7x^2 - 10x + 5 = 0$
ⓒ $25x^2 - 90x + 81 = 0$
ⓓ $15x^2 - 8x + 4 = 0$

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

453.

ⓐ $16r^2 - 8r + 1 = 0$
ⓑ $5t^2 - 8t + 3 = 9$
ⓒ $3(c + 2)^2 = 15$

454.

ⓐ $4d^2 + 10d - 5 = 21$
ⓑ $25x^2 - 60x + 36 = 0$
ⓒ $6(5v - 7)^2 = 150$

9.4 Solve Equations in Quadratic Form

Solve Equations in Quadratic Form

In the following exercises, solve.

455. $x^4 - 14x^2 + 24 = 0$

456. $x^4 + 4x^2 - 32 = 0$

457. $4x^4 - 5x^2 + 1 = 0$

458. $(2y + 3)^2 + 3(2y + 3) - 28 = 0$

459. $x + 3\sqrt{x} - 28 = 0$

460. $6x + 5\sqrt{x} - 6 = 0$

461. $x^{\frac{2}{3}} - 10x^{\frac{1}{3}} + 24 = 0$

462. $x + 7x^{\frac{1}{2}} + 6 = 0$

463. $8x^{-2} - 2x^{-1} - 3 = 0$

9.5 Solve Applications of Quadratic Equations

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

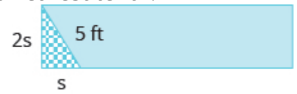
464. Find two consecutive odd numbers whose product is 323.

465. Find two consecutive even numbers whose product is 624.

466. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.

467. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.

468. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.



469. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

470. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

471. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

472. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

473. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

474. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

475. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

9.6 Graph Quadratic Functions Using Properties

Recognize the Graph of a Quadratic Function

In the following exercises, graph by plotting point.

476. Graph $y = x^2 - 2$

477. Graph $y = -x^2 + 3$

In the following exercises, determine if the following parabolas open up or down.

478.

ⓐ $y = -3x^2 + 3x - 1$

ⓑ $y = 5x^2 + 6x + 3$

479.

ⓐ $y = x^2 + 8x - 1$

ⓑ $y = -4x^2 - 7x + 1$

Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find ⓐ the equation of the axis of symmetry and ⓑ the vertex.

480. $y = -x^2 + 6x + 8$

481. $y = 2x^2 - 8x + 1$

Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

482. $y = x^2 - 4x + 5$

483. $y = x^2 - 8x + 15$

484. $y = x^2 - 4x + 10$

485. $y = -5x^2 - 30x - 46$

486. $y = 16x^2 - 8x + 1$

487. $y = x^2 + 16x + 64$

Graph Quadratic Functions Using Properties

In the following exercises, graph by using its properties.

488. $y = x^2 + 8x + 15$

489. $y = x^2 - 2x - 3$

490. $y = -x^2 + 8x - 16$

491. $y = 4x^2 - 4x + 1$

492. $y = x^2 + 6x + 13$

493. $y = -2x^2 - 8x - 12$

Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

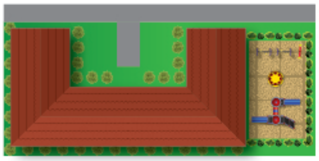
494. $y = 7x^2 + 14x + 6$

495. $y = -3x^2 + 12x - 10$

In the following exercises, solve. Rounding answers to the nearest tenth.

496. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.

497. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.



9.7 Graph Quadratic Functions Using Transformations

Graph Quadratic Functions of the form $f(x) = x^2 + k$

In the following exercises, graph each function using a vertical shift.

498. $g(x) = x^2 + 4$

499. $h(x) = x^2 - 3$

In the following exercises, graph each function using a horizontal shift.

500. $f(x) = (x + 1)^2$

501. $g(x) = (x - 3)^2$

In the following exercises, graph each function using transformations.

502. $f(x) = (x + 2)^2 + 3$

503. $f(x) = (x + 3)^2 - 2$

504. $f(x) = (x - 1)^2 + 4$

505. $f(x) = (x - 4)^2 - 3$

Graph Quadratic Functions of the form $f(x) = ax^2$

In the following exercises, graph each function.

506. $f(x) = 2x^2$

507. $f(x) = -x^2$

508. $f(x) = \frac{1}{2}x^2$

Graph Quadratic Functions Using Transformations

In the following exercises, rewrite each function in the $f(x) = a(x - h)^2 + k$ form by completing the square.

509. $f(x) = 2x^2 - 4x - 4$

510. $f(x) = 3x^2 + 12x + 8$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x - h)^2 + k$ form and ⓑ graph it by using transformations.

511. $f(x) = 3x^2 - 6x - 1$

512. $f(x) = -2x^2 - 12x - 5$

513. $f(x) = 2x^2 + 4x + 6$

514. $f(x) = 3x^2 - 12x + 7$

In the following exercises, ⓐ rewrite each function in $f(x) = a(x - h)^2 + k$ form and ⓑ graph it using properties.

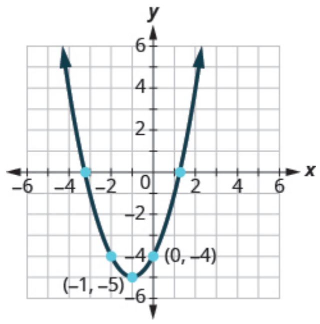
515. $f(x) = -3x^2 - 12x - 5$

516. $f(x) = 2x^2 - 12x + 7$

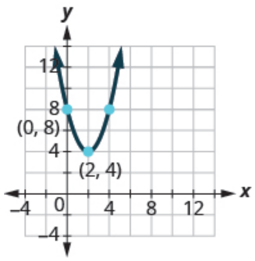
Find a Quadratic Function from its Graph

In the following exercises, write the quadratic function in $f(x) = a(x - h)^2 + k$ form.

517.



518.



9.8 Solve Quadratic Inequalities

Solve Quadratic Inequalities Graphically

In the following exercises, solve graphically and write the solution in interval notation.

519. $x^2 - x - 6 > 0$

520. $x^2 + 4x + 3 \leq 0$

521. $-x^2 - x + 2 \geq 0$

522. $-x^2 + 2x + 3 < 0$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

523. $x^2 - 6x + 8 < 0$

524. $x^2 + x > 12$

525. $x^2 - 6x + 4 \leq 0$

526. $2x^2 + 7x - 4 > 0$

527. $-x^2 + x - 6 > 0$

528. $x^2 - 2x + 4 \geq 0$

PRACTICE TEST

529. Use the Square Root Property to solve the quadratic equation $3(w+5)^2 = 27$.

530. Use Completing the Square to solve the quadratic equation $a^2 - 8a + 7 = 23$.

531. Use the Quadratic Formula to solve the quadratic equation $2m^2 - 5m + 3 = 0$.

Solve the following quadratic equations. Use any method.

532. $2x(3x-2) - 1 = 0$

533. $\frac{9}{4}y^2 - 3y + 1 = 0$

Use the discriminant to determine the number and type of solutions of each quadratic equation.

534. $6p^2 - 13p + 7 = 0$

535. $3q^2 - 10q + 12 = 0$

Solve each equation.

536. $4x^4 - 17x^2 + 4 = 0$

537. $y^{\frac{2}{3}} + 2y^{\frac{1}{3}} - 3 = 0$

For each parabola, find ⓐ which direction it opens, ⓑ the equation of the axis of symmetry, ⓒ the vertex, ⓓ the x- and y-intercepts, and e) the maximum or minimum value.

538. $y = 3x^2 + 6x + 8$

539. $y = -x^2 - 8x + 16$

Graph each quadratic function using intercepts, the vertex, and the equation of the axis of symmetry.

540. $f(x) = x^2 + 6x + 9$

541. $f(x) = -2x^2 + 8x + 4$

In the following exercises, graph each function using transformations.

542. $f(x) = (x+3)^2 + 2$

543. $f(x) = x^2 - 4x - 1$

In the following exercises, solve each inequality algebraically and write any solution in interval notation.

544. $x^2 - 6x - 8 \leq 0$

545. $2x^2 + x - 10 > 0$

Model the situation with a quadratic equation and solve by any method.

546. Find two consecutive even numbers whose product is 360.

547. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

548. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h = -16t^2 + 86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

LEARNING OBJECTIVES

In this section you will:

- Use interval notation.
- Use properties of inequalities.
- Solve inequalities in one variable algebraically.
- Solve absolute value inequalities.

2.7 LINEAR INEQUALITIES AND ABSOLUTE VALUE INEQUALITIES

Figure 1

It is not easy to make the honor role at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

Using Interval Notation

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.

We can use a number line as shown in **Figure 2**. The blue ray begins at $x = 4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.

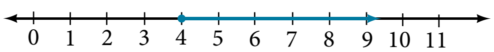


Figure 2

We can use set-builder notation: $\{x|x \geq 4\}$, which translates to "all real numbers x such that x is greater than or equal to 4." Notice that braces are used to indicate a set.

The third method is **interval notation**, in which solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method, as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an **interval**, or a set of numbers in which a solution falls, are $[-2, 6)$, or all numbers between -2 and 6, including -2, but not including 6; $(-1, 0)$, all real numbers between, but not including -1 and 0; and $(-\infty, 1]$, all real numbers less than and including 1. **Table 1** outlines the possibilities.

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers between a and b, but not including a or b	$\{x \| a < x < b\}$	(a, b)
All real numbers greater than a, but not including a	$\{x \| x > a\}$	(a, ∞)
All real numbers less than b, but not including b	$\{x \| x < b\}$	$(-\infty, b)$
All real numbers greater than a, including a	$\{x \| x \geq a\}$	$[a, \infty)$

Set Indicated	Set-Builder Notation	Interval Notation
All real numbers less than b, including b	$\{x\mid x \le b\}$	$(-\infty, b]$
All real numbers between a and b, including a	$\{x\mid a \le x < b\}$	$[a, b)$
All real numbers between a and b, including b	$\{x\mid a < x \le b\}$	$(a, b]$
All real numbers between a and b, including a and b	$\{x\mid a \le x \le b\}$	$[a, b]$
All real numbers less than a or greater than b	$\{x\mid x < a \text{ and } x > b\}$	$(-\infty, a) \cup (b, \infty)$
All real numbers	$\{x\mid x \text{ is all real numbers}\}$	$(-\infty, \infty)$

Table 1

Example 1 Using Interval Notation to Express All Real Numbers Greater Than or Equal to *a*

Use interval notation to indicate all real numbers greater than or equal to -2.

Solution Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

Try It #1

Use interval notation to indicate all real numbers between and including -3 and 5.

Example 2 Using Interval Notation to Express All Real Numbers Less Than or Equal to *a* or Greater Than or Equal to *b*

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1.

Solution We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1. So, this interval begins at $-\infty$ and ends at -1, which is written as $(-\infty, -1]$.

The second interval must show all real numbers greater than or equal to 1, which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, $\cup$, between the two intervals.

$$(-\infty, -1] \cup [1, \infty)$$

Try It #2

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.

Using the Properties of Inequalities

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equalities. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

properties of inequalities

Addition Property	If $a < b$, then $a + c < b + c$.
Multiplication Property	If $a < b$ and $c > 0$, then $ac < bc$.
	If $a < b$ and $c < 0$, then $ac > bc$.

These properties also apply to $a \le b$, $a > b$, and $a \ge b$.

Example 3 Demonstrating the Addition Property

Illustrate the addition property for inequalities by solving each of the following:

a. $x - 15 < 4$ **b.** $6 \ge x - 1$ **c.** $x + 7 > 9$

Solution The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.
$$x - 15 < 4$$
$$x - 15 + 15 < 4 + 15 \quad \text{Add 15 to both sides.}$$
$$x < 19$$

b.
$$6 \geq x - 1$$
$$6 + 1 \geq x - 1 + 1 \quad \text{Add 1 to both sides.}$$
$$7 \geq x$$

c.
$$x + 7 > 9$$
$$x + 7 - 7 > 9 - 7 \quad \text{Subtract 7 from both sides.}$$
$$x > 2$$

Try It #3

Solve: $3x - 2 < 1$.

Example 4 Demonstrating the Multiplication Property

Illustrate the multiplication property for inequalities by solving each of the following:

a. $3x < 6$ **b.** $-2x - 1 \geq 5$ **c.** $5 - x > 10$

Solution

a.
$$3x < 6$$
$$\frac{1}{3}(3x) < (6)\frac{1}{3}$$
$$x < 2$$

b.
$$-2x - 1 \geq 5$$
$$-2x \geq 6$$
$$\left(-\frac{1}{2}\right)(-2x) \geq (6)\left(-\frac{1}{2}\right) \quad \text{Multiply by } -\frac{1}{2}.$$
$$x \leq -3 \quad \text{Reverse the inequality.}$$

c.
$$5 - x > 10$$
$$-x > 5$$
$$(-1)(-x) > (5)(-1) \quad \text{Multiply by } -1.$$
$$x < -5 \quad \text{Reverse the inequality.}$$

Try It #4

Solve: $4x + 7 \geq 2x - 3$.

Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

Example 5 **Solving an Inequality Algebraically**

Solve the inequality: $13 - 7x \geq 10x - 4$.

Solution Solving this inequality is similar to solving an equation up until the last step.

$$13 - 7x \geq 10x - 4$$

$$13 - 17x \geq -4 \qquad \text{Move variable terms to one side of the inequality.}$$

$$-17x \geq -17 \qquad \text{Isolate the variable term.}$$

$$x \leq 1 \qquad \text{Dividing both sides by } -17 \text{ reverses the inequality.}$$

The solution set is given by the interval $(-\infty, 1]$, or all real numbers less than and including 1.

Try It #5

Solve the inequality and write the answer using interval notation: $-x + 4 < \frac{1}{2}x + 1$.

Example 6 **Solving an Inequality with Fractions**

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$.

Solution We begin solving in the same way we do when solving an equation.

$$-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x.$$

$$-\frac{3}{4}x - \frac{2}{3}x \geq -\frac{5}{8} \qquad \text{Put variable terms on one side.}$$

$$-\frac{9}{12}x - \frac{8}{12}x \geq -\frac{5}{8} \qquad \text{Write fractions with common denominator.}$$

$$-\frac{17}{12}x \geq -\frac{5}{8}$$

$$x \leq -\frac{5}{8}\left(-\frac{12}{17}\right) \qquad \text{Multiplying by a negative number reverses the inequality.}$$

$$x \leq \frac{15}{34}$$

The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

Try It #6

Solve the inequality and write the answer in interval notation: $-\frac{5}{6}x \leq \frac{3}{4} + \frac{8}{3}x$.

Understanding Compound Inequalities

A **compound inequality** includes two inequalities in one statement. A statement such as $4 < x \leq 6$ means $4 < x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

Example 7 **Solving a Compound Inequality**

Solve the compound inequality: $3 \leq 2x + 2 < 6$.

Solution The first method is to write two separate inequalities: $3 \leq 2x + 2$ and $2x + 2 < 6$. We solve them independently.

$$3 \leq 2x + 2 \qquad \text{and} \qquad 2x + 2 < 6$$

$$1 \leq 2x \qquad\qquad 2x < 4$$

$$\frac{1}{2} \leq x \qquad\qquad x < 2$$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$\frac{1}{2} \leq x < 2$$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.

The second method is to leave the compound inequality intact, and perform solving procedures on the three parts at the same time.

$$3 \leq 2x + 2 < 6$$

$$1 \leq 2x < 4 \qquad \text{Isolate the variable term, and subtract 2 from all three parts.}$$

$$\frac{1}{2} \leq x < 2 \qquad \text{Divide through all three parts by 2.}$$

We get the same solution: $\left[\frac{1}{2}, 2\right)$.

Try It #7

Solve the compound inequality: $4 < 2x - 8 \leq 10$.

Example 8 Solving a Compound Inequality with the Variable in All Three Parts

Solve the compound inequality with variables in all three parts: $3 + x > 7x - 2 > 5x - 10$.

Solution Let's try the first method. Write two inequalities:

$$\begin{aligned} 3 + x &> 7x - 2 & \quad \text{and} \quad && 7x - 2 &> 5x - 10 \\ 3 &> 6x - 2 & && 2x - 2 &> -10 \\ 5 &> 6x & && 2x &> -8 \\ \frac{5}{6} &> x & && x &> -4 \\ x &< \frac{5}{6} & && -4 &< x \end{aligned}$$

The solution set is $-4 < x < \frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right, as they appear on a number line. See **Figure 3**.

Figure 3

Try It #8

Solve the compound inequality: $3y < 4 - 5y < 5 + 3y$.

Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of x, as it is x units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$|A| < B, |A| \leq B, |A| > B, \text{ or } |A| \geq B,$$

Where A, and sometimes B, represents an algebraic expression dependent on a variable x. Solving the inequality means finding the set of all x -values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations. The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within \$200 of \$600. We can solve algebraically for the set of x-values such that the distance between x and 600 is less than or equal to 200. We represent the distance between x and 600 as $|x - 600|$, and therefore, $|x - 600| \leq 200$ or

$$-200 \leq x - 600 \leq 200$$
$$-200 + 600 \leq x - 600 + 600 \leq 200 + 600$$
$$400 \leq x \leq 800$$

This means our returns would be between \$400 and \$800.

To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

> ***absolute value inequalities***
> For an algebraic expression X, and $k > 0$, an **absolute value inequality** is an inequality of the form
>
> $$|X| < k \text{ is equivalent to } -k < X < k$$
> $$|X| > k \text{ is equivalent to } X < -k \text{ or } X > k$$
>
> These statements also apply to $|X| \leq k$ and $|X| \geq k$.

Example 9 Determining a Number within a Prescribed Distance

Describe all values x within a distance of 4 from the number 5.

Solution We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as in **Figure 4**, to represent the condition to be satisfied.

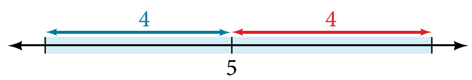


Figure 4

The distance from x to 5 can be represented using an absolute value symbol, $|x - 5|$. Write the values of x that satisfy the condition as an absolute value inequality.

$$|x - 5| \leq 4$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$x - 5 \leq 4 \qquad \text{and} \qquad x - 5 \geq -4$$
$$x \leq 9 \qquad\qquad\qquad x \geq 1$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9.

So $|x - 5| \leq 4$ is equivalent to $[1, 9]$ in interval notation.

Try It #9
Describe all x-values within a distance of 3 from the number 2.

Example 10 Solving an Absolute Value Inequality

Solve $|x - 1| \leq 3$.

Solution

$$|x - 1| \leq 3$$
$$-3 \leq x - 1 \leq 3$$
$$-2 \leq x \leq 4$$
$$[-2, 4]$$

Example 11 Using a Graphical Approach to Solve Absolute Value Inequalities

Given the equation $y = -\frac{1}{2}|4x - 5| + 3$, determine the x-values for which the y-values are negative.

Solution We are trying to determine where $y < 0$, which is when $-\frac{1}{2}|4x - 5| + 3 < 0$. We begin by isolating the absolute value.

$$-\frac{1}{2}|4x - 5| < -3 \qquad \text{Multiply both sides by –2, and reverse the inequality.}$$

$$|4x - 5| > 6$$

Next, we solve for the equality $|4x - 5| = 6$.

$$4x - 5 = 6 \qquad \text{or} \qquad 4x - 5 = -6$$

$$4x = 11 \qquad\qquad 4x = -1$$

$$x = \frac{11}{4} \qquad\qquad x = -\frac{1}{4}$$

Now, we can examine the graph to observe where the y-values are negative. We observe where the branches are below the x-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = -\frac{1}{4}$ and $x = \frac{11}{4}$, and that the graph opens downward. See **Figure 5**.

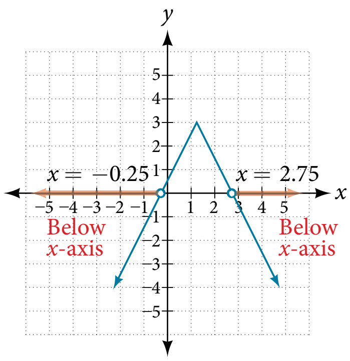


Figure 5

Try It #10

Solve $-2|k - 4| \leq -6$.

Access these online resources for additional instruction and practice with linear inequalities and absolute value inequalities.

- Interval Notation (http://openstaxcollege.org/l/intervalnotn)
- How to Solve Linear Inequalities (http://openstaxcollege.org/l/solvelinineq)
- How to Solve an Inequality (http://openstaxcollege.org/l/solveineq)
- Absolute Value Equations (http://openstaxcollege.org/l/absvaleq)
- Compound Inequalities (http://openstaxcollege.org/l/compndineqs)
- Absolute Value Inequalities (http://openstaxcollege.org/l/absvalineqs)

2.7 SECTION EXERCISES

VERBAL

1. When solving an inequality, explain what happened from Step 1 to Step 2:
Step 1 $\quad -2x > 6$
Step 2 $\quad x < -3$

2. When solving an inequality, we arrive at:
$x + 2 < x + 3$
$2 < 3$
Explain what our solution set is.

3. When writing our solution in interval notation, how do we represent all the real numbers?

4. When solving an inequality, we arrive at:
$x + 2 > x + 3$
$2 > 3$
Explain what our solution set is.

5. Describe how to graph $y = |x - 3|$

ALGEBRAIC

For the following exercises, solve the inequality. Write your final answer in interval notation

6. $4x - 7 \leq 9$

7. $3x + 2 \geq 7x - 1$

8. $-2x + 3 > x - 5$

9. $4(x + 3) \geq 2x - 1$

10. $-\frac{1}{2}x \leq -\frac{5}{4} + \frac{2}{5}x$

11. $-5(x - 1) + 3 > 3x - 4 - 4x$

12. $-3(2x + 1) > -2(x + 4)$

13. $\frac{x+3}{8} - \frac{x+5}{5} \geq \frac{3}{10}$

14. $\frac{x-1}{3} + \frac{x+2}{5} \leq \frac{3}{5}$

For the following exercises, solve the inequality involving absolute value. Write your final answer in interval notation.

15. $|x + 9| \geq -6$

16. $|2x + 3| < 7$

17. $|3x - 1| > 11$

18. $|2x + 1| + 1 \leq 6$

19. $|x - 2| + 4 \geq 10$

20. $|-2x + 7| \leq 13$

21. $|x - 7| < -4$

22. $|x - 20| > -1$

23. $\left|\frac{x-3}{4}\right| < 2$

For the following exercises, describe all the x-values within or including a distance of the given values.

24. Distance of 5 units from the number 7

25. Distance of 3 units from the number 9

26. Distance of 10 units from the number 4

27. Distance of 11 units from the number 1

For the following exercises, solve the compound inequality. Express your answer using inequality signs, and then write your answer using interval notation.

28. $-4 < 3x + 2 \leq 18$

29. $3x + 1 > 2x - 5 > x - 7$

30. $3y < 5 - 2y < 7 + y$

31. $2x - 5 < -11$ or $5x + 1 \geq 6$

32. $x + 7 < x + 2$

GRAPHICAL

For the following exercises, graph the function. Observe the points of intersection and shade the x-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.

33. $|x - 1| > 2$ **34.** $|x + 3| \geq 5$ **35.** $|x + 7| \leq 4$ **36.** $|x - 2| < 7$ **37.** $|x - 2| < 0$

For the following exercises, graph both straight lines (left-hand side being y_1 and right-hand side being y_2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the y-values of the lines.

38. $x + 3 < 3x - 4$ **39.** $x - 2 > 2x + 1$ **40.** $x + 1 > x + 4$ **41.** $\frac{1}{2}x + 1 > \frac{1}{2}x - 5$

42. $4x + 1 < \frac{1}{2}x + 3$

NUMERIC

For the following exercises, write the set in interval notation.

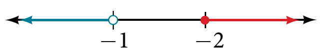
43. $\{x|-1 < x < 3\}$ **44.** $\{x|x \geq 7\}$ **45.** $\{x|x < 4\}$ **46.** $\{x|\ x$ is all real numbers$\}$

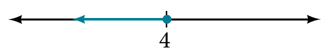
For the following exercises, write the interval in set-builder notation.

47. $(-\infty, 6)$ **48.** $(4, \infty)$ **49.** $[-3, 5)$ **50.** $[-4, 1] \cup [9, \infty)$

For the following exercises, write the set of numbers represented on the number line in interval notation.

51.


52.


53.


TECHNOLOGY

For the following exercises, input the left-hand side of the inequality as a Y1 graph in your graphing utility. Enter **Y2=** the right-hand side. Entering the absolute value of an expression is found in the **MATH** menu, **Num**, **1:abs(**. Find the points of intersection, recall (**2nd CALC 5:intersection**, **1st curve, enter, 2nd curve, enter, guess, enter**). Copy a sketch of the graph and shade the x-axis for your solution set to the inequality. Write final answers in interval notation.

54. $|x + 2| - 5 < 2$ **55.** $-\frac{1}{2}|x + 2| < 4$ **56.** $|4x + 1| - 3 > 2$ **57.** $|x - 4| < 3$

58. $|x + 2| \geq 5$

EXTENSIONS

59. Solve $|3x + 1| = |2x + 3|$

60. Solve $x^2 - x > 12$

61. $\frac{x - 5}{x + 7} \leq 0, x \neq -7$

62. $p = -x^2 + 130x - 3{,}000$ is a profit formula for a small business. Find the set of x-values that will keep this profit positive.

REAL-WORLD APPLICATIONS

63. In chemistry the volume for a certain gas is given by $V = 20T$, where V is measured in cc and T is temperature in ºC. If the temperature varies between 80ºC and 120ºC, find the set of volume values.

64. A basic cellular package costs \$20/mo. for 60 min of calling, with an additional charge of \$0.30/min beyond that time. The cost formula would be $C = \$20 + .30(x - 60)$. If you have to keep your bill lower than \$50, what is the maximum calling minutes you can use?

2.8 Graphing Systems of Linear Inequalities

Determine whether an ordered pair is a solution of a system of linear inequalities

The definition of a **system of linear inequalities** is very similar to the definition of a system of linear equations.

> **System of Linear Inequalities**
>
> Two or more linear inequalities grouped together form a system of linear inequalities.

A system of linear inequalities looks like a system of linear equations, but it has inequalities instead of equations. A system of two linear inequalities is shown here.

$$\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$$

To solve a system of linear inequalities, we will find values of the variables that are solutions to both inequalities. We solve the system by using the graphs of each inequality and show the solution as a graph. We will find the region on the plane that contains all ordered pairs (x, y) that make both inequalities true.

> **Solutions of a System of Linear Inequalities**
>
> Solutions of a system of linear inequalities are the values of the variables that make all the inequalities true.
>
> The solution of a system of linear inequalities is shown as a shaded region in the x, y coordinate system that includes all the points whose ordered pairs make the inequalities true.

To determine if an ordered pair is a solution to a system of two inequalities, we substitute the values of the variables into each inequality. If the ordered pair makes both inequalities true, it is a solution to the system.

Example 1 Determine whether the ordered pair is a solution to the system $\begin{cases} x + 4y \geq 10 \\ 3x - 2y < 12 \end{cases}$

a. $(-2, 4)$ b. $(3, 1)$

Solution:
a. Is the ordered pair $(-2, 4)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$$\begin{aligned} x + 4y &\geq 10 \\ -2 + 4 \cdot 4 &\geq 10 \\ 14 &\geq 10 \text{ true} \end{aligned} \qquad \begin{aligned} 3x - 2y &< 12 \\ 3(-2) - 2 \cdot 4 &< 12 \\ -14 &< 12 \text{ true} \end{aligned}$$

The ordered pair $(-2, 4)$ made both inequalities true. Therefore $(-2, 4)$ is a solution to this system.

b. Is the ordered pair $(3, 1)$ a solution?

We substitute $x = -2$ and $y = 4$ into both inequalities.

$$\begin{aligned} x + 4y \geq&\ 10 \\ 3 + 4 \cdot 1 \geq&\ 10 \\ 7 \geq&\ 10 \text{ false} \end{aligned} \qquad\qquad \begin{aligned} 3x - 2y <&\ 12 \\ 3 \cdot 3 - 2 \cdot 1 <&\ 12 \\ 7 <&\ 12 \text{ true} \end{aligned}$$

The ordered pair $(3, 1)$ made one inequality true, but the other one false. Therefore $(3, 1)$ is **not** a solution to this system.

Try It #1: Determine whether the ordered pair is a solution to the system: $\begin{cases} x - 5y > 10 \\ 2x + 3y > -2 \end{cases}$

a. $(3, -1)$ b. $(6, -3)$

Try It #2: Determine whether the ordered pair is a solution to the system: $\begin{cases} y > 4x - 2 \\ 4x - 2y > 20 \end{cases}$

a. $(-2, 1)$ b. $(4, -1)$

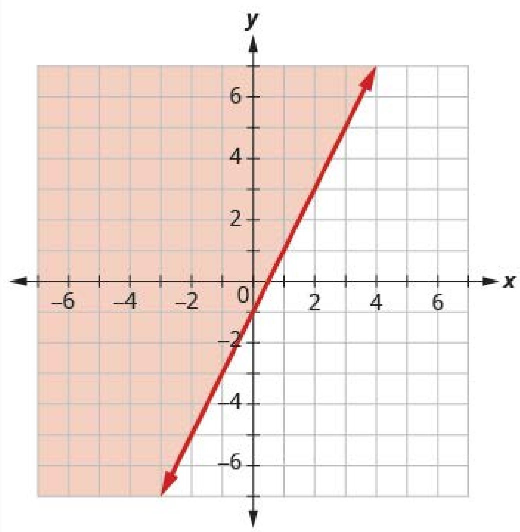
Solve a System of Linear Inequalities by Graphing

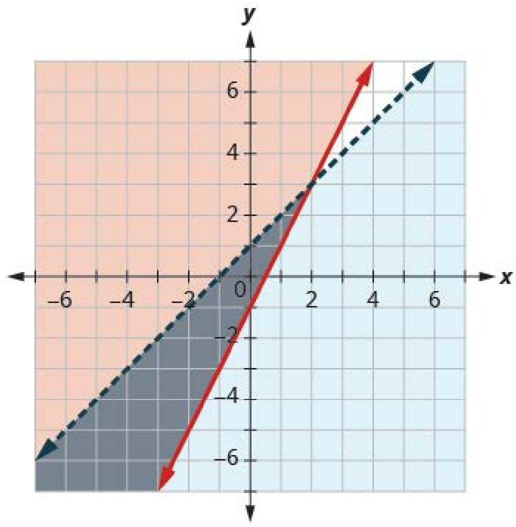
The solution to a single linear inequality is the region on one side of the boundary line that contains all the points that make the inequality true. The solution to a system of two linear inequalities is a region that contains the solutions to both inequalities. To find this region, we will graph each inequality separately and then locate the region where they are both true. The solution is always shown as a graph.

Example 2 HOW TO SOLVE A SYSTEM OF LINEAR INEQUALITIES BY GRAPHING

Solve the system by graphing: $\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$

Solution:

Step 1. Graph the first inequality. • Graph the boundary line. • Shade in the side of the boundary line where the inequality is true.	We will graph $y \geq 2x - 1$. We graph the line $y = 2x - 1$. It is a solid line because the inequality sign is $\geq$. We choose (0, 0) as a test point. It is a solution to $y \geq 2x - 1$, so we shade in above the boundary line.	$\begin{cases} y \geq 2x - 1 \\ y < x + 1 \end{cases}$ 

Step 2. On the same grid, graph the second inequality. • Graph the boundary line. • Shade in the side of that boundary line where the inequality is true.	We will graph $y < x + 1$ on the same grid. We graph the line $y = x + 1$. It is a dashed line because the inequality sign is $<$. Again, we use (0, 0) as a test point. It is a solution so we shade in that side of the line $y = x + 1$.	
Step 3. The solution is the region where the shading overlaps.	The point where the boundary lines intersect is not a solution because it is not a solution to $y < x + 1$.	The solution is all points in the area shaded twice—which appears as the darkest shaded region.
Step 4. Check by choosing a test point.	We'll use (–1, –1) as a test point.	Is (–1, –1) a solution to $y \geq 2x - 1$? $-1 \overset{?}{\geq} 2(-1) - 1$ $-1 \geq -3$ true Is (–1, –1) a solution to $y < x + 1$? $-1 \overset{?}{<} -1 + 1$ $-1 < 0$ true The region containing (–1, –1) is the solution to this system.

Try It #3: Solve the system by graphing: $\begin{cases} y < 3x - 2 \\ y > -x - 1 \end{cases}$

Try It #4: Solve the system by graphing: $\begin{cases} y < -\dfrac{1}{2}x + 3 \\ y < 3x - 4 \end{cases}$

HOW TO SOLVE A SYSTEM OF LINEAR INEQUALITIES BY GRAPHING.

Step 1: Graph the first inequality.

- Graph the boundary line.
- Shade in the side of the boundary line where the inequality is true.

Step 2: On the same grid, graph the second inequality.

- Graph the boundary line.
- Shade in the side of that boundary line where the inequality is true.

Step 3: The solution is the region where the shading overlaps.

Step 4: Check by choosing a test point.

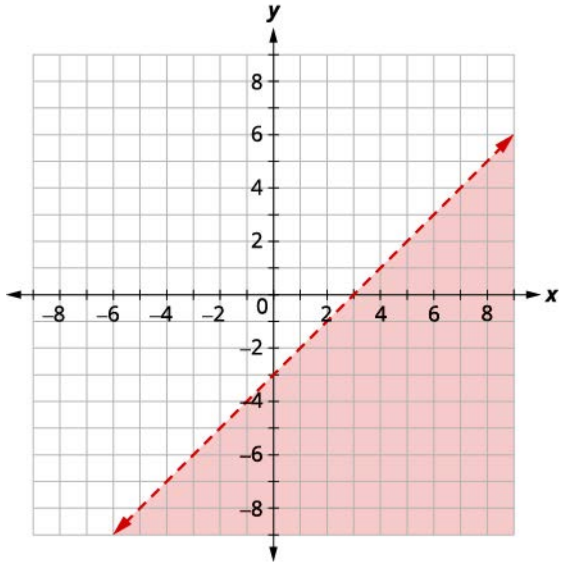
Example 3 Solve the system by graphing: $\begin{cases} x - y > 3 \\ y < -\frac{1}{5}x + 4 \end{cases}$

Solution:

Graph $x - y > 3$, by graphing $x - y = 3$ and testing a point.

The intercepts are $x = 3$ and $y = -3$ and the boundary line will be dashed.

Test $(0, 0)$ which makes the inequality false, so shade (red) the side that does not contain $(0, 0)$.

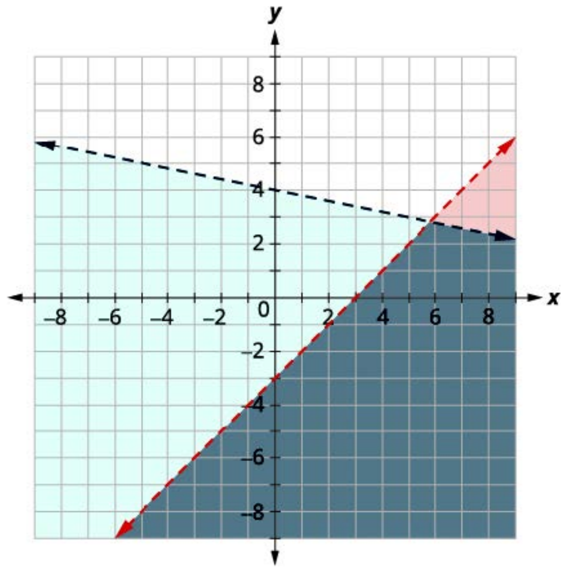


Graph $y < -\frac{1}{5}x + 4$, by graphing $y = -\frac{1}{5}x + 43$ using the slope $m = -\frac{1}{5}$ and y-intercept $b = 4$.

The boundary line will be dashed.

Test $(0, 0)$ which makes the inequality true, so shade (blue) the side that not contains $(0, 0)$.

Choose a test point in the solution and verify that it is a solution to both inequalities.



The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twicewhich appears as the darkest shaded region.

Try It #5: Solve the system by graphing: $\begin{cases} x + y \leq 2 \\ y \geq \frac{2}{3}x - 1 \end{cases}$

Try It #6: Solve the system by graphing: $\begin{cases} 3x - 2y \leq 6 \\ y > \frac{1}{4}x - +5 \end{cases}$

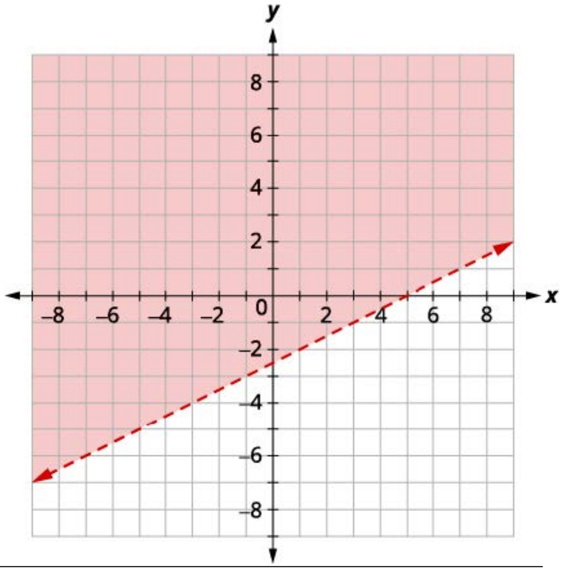
Example 4 Solve the system by graphing: $\begin{cases} x - 2y < 5 \\ y > -4 \end{cases}$

Solution:

Graph $x - 2y < 5$, by graphing $x - 2y = 5$ and testing a point.

The intercepts are $x = 5$ and $y = -2.5$ and the boundary line will be dashed.

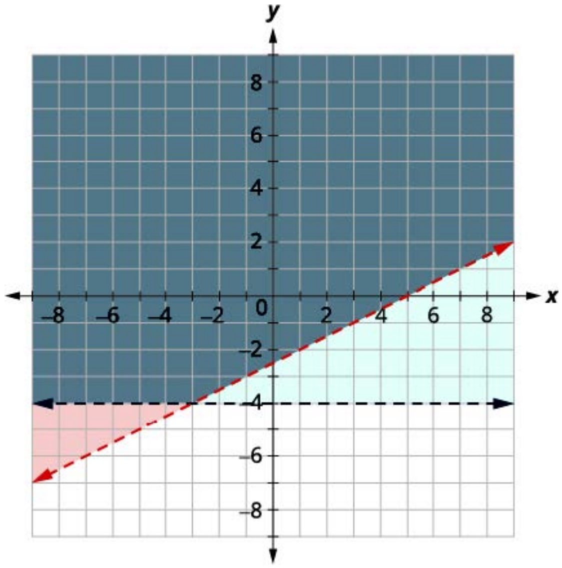
Test $(0, 0)$ which makes the inequality true, so shade (red) the side that contains $(0, 0)$.



Graph $y > -4$, by graphing $y = -4$ and recognizing that it is a horizontal line through $y = -4$.

The boundary line will be dashed.

Test $(0, 0)$ which makes the inequality true, so shade (blue) the side that contains $(0, 0)$.



The point $(0, 0)$ is in the solution and we have already found it to be a solution of each inequality. The point of intersection of the two lines is not included as both boundary lines were dashed. The solution is the area shaded twice which appears as the darkest shaded region.

Try It #7: Solve the system by graphing: $\begin{cases} y \geq 3x - 2 \\ y < -1 \end{cases}$

Try It #8: Solve the system by graphing: $\begin{cases} x > -4 \\ x - 2y \geq -4 \end{cases}$

Systems of linear inequalities where the boundary lines are parallel might have no solution.

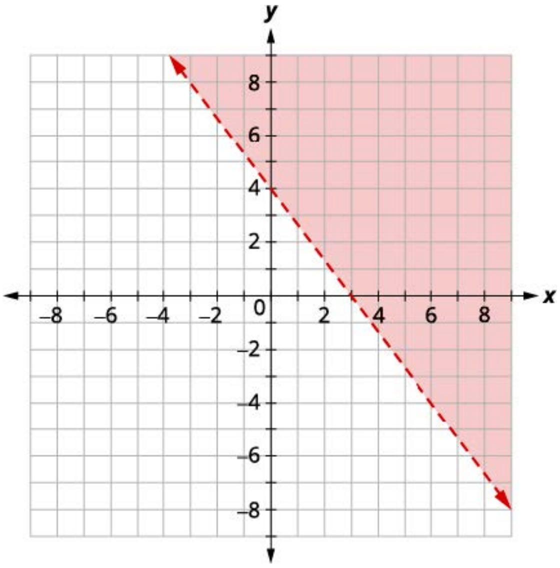
Example 5 Solve the system by graphing: $\begin{cases} 4x + 3y \geq 12 \\ y < -\dfrac{4}{3}x + 1 \end{cases}$

Solution:

Graph $4x + 3y \geq 12$, by graphing $4x + 3y = 12$ and testing a point.

The intercepts are $x = 3$ and $y = 4$ and the boundary line will be solid.

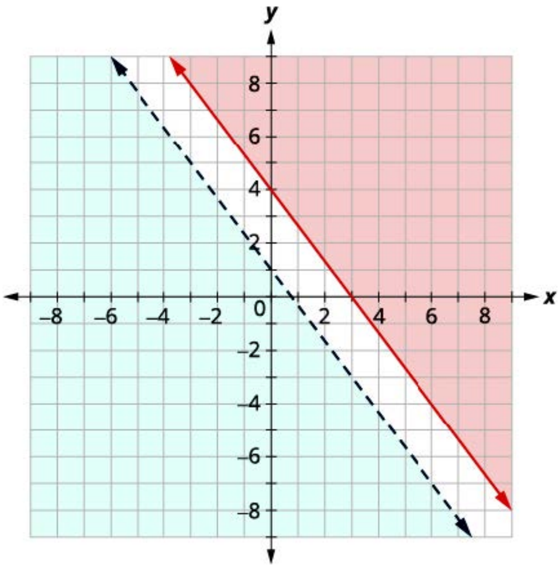
Test $(0, 0)$ which makes the inequality false, so shade (red) the side that does not contain $(0, 0)$.



Graph $y < -\dfrac{4}{3}x + 1$, by graphing $y = -\dfrac{4}{3}x + 1$ using the slope $m = -\dfrac{4}{3}$ and y-intercept $b = 1$.

The boundary line will be dashed.

Test $(0, 0)$ which makes the inequality true, so shade (blue) the side that contains $(0, 0)$.



There is no point in both shaded regions, so the system has no solution.

Try It #9: Solve the system by graphing: $\begin{cases} 3x - 2y \geq 12 \\ y \geq \dfrac{3}{2}x + 1 \end{cases}$

Try It #10: Solve the system by graphing: $\begin{cases} x + 3y > 8 \\ y < -\dfrac{1}{3}x - 2 \end{cases}$

Some systems of linear inequalities where the boundary lines are parallel will have a solution.

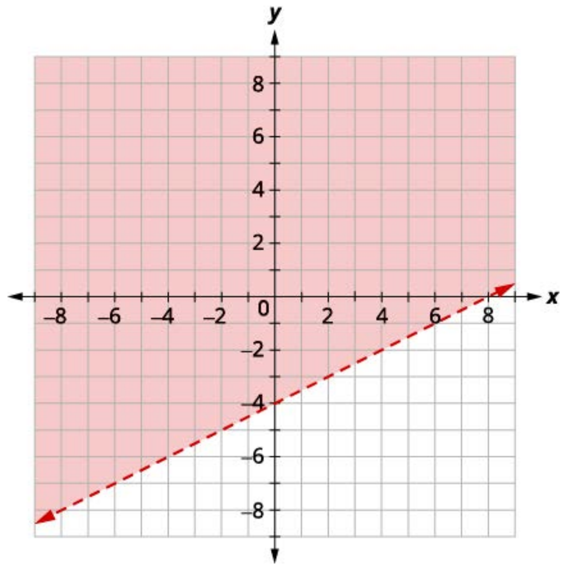
Example 6 Solve the system by graphing: $\begin{cases} y > \frac{1}{2}x - 4 \\ x - 2y < -4 \end{cases}$

Solution:

$y > \frac{1}{2}x - 4$, by graphing $y = \frac{1}{2}x - 4$ using the slope $m = \frac{1}{2}$ and y-intercept $b = -4$.

The boundary line will be dashed.

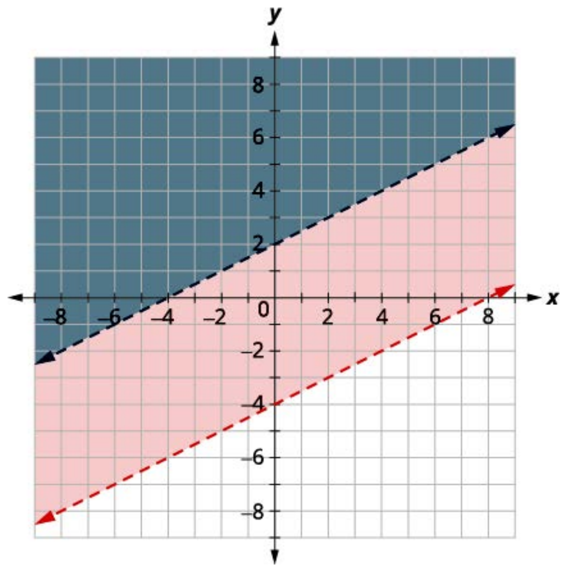
Test $(0, 0)$ which makes the inequality true, so shade (red) the side that contains $(0, 0)$.



Graph $x - 2y < -4$, by graphing $x - 2y = -4$ and testing a point.

The intercepts are $x = -4$ and $y = 2$ and the boundary line will be dashed.

Test $(0, 0)$ which makes the inequality false, so shade (blue) the side that contains $(0, 0)$.



No point on the boundary lines is included in the solution as both lines are dashed. The solution is the region that is shaded twice which is also the solution to $x - 2y < -4$.

Choose a test point in the solution and verify that it is a solution to both inequalities.

Try It #11: Solve the system by graphing: $\begin{cases} y \geq 3x + 1 \\ -3x + y \geq -4 \end{cases}$

Try It #12: Solve the system by graphing: $\begin{cases} y < -\frac{1}{4}x + 2 \\ x + 4y \geq 4 \end{cases}$

Solve Applications of Systems of Inequalities

The first thing we'll need to do to solve applications of systems of inequalities is to translate each condition into an inequality. Then we graph the system, as we did above, to see the region that contains the solutions. Many situations will be realistic only if both variables are positive, so we add inequalities to the system as additional requirements.

Example 7 Christy sells her photographs at a booth at a street fair. At the start of the day, she wants to have at least 25 photos to display at her booth. Each small photo she displays costs her \$4 and each large photo costs her \$10. She doesn't want to spend more than \$200 on photos to display.

(a) Write a system of inequalities to model this situation.

(b) Graph the system.

(c) Could she display 10 small and 20 large photos?

(d) Could she display 20 large and 10 small photos?

Solution:

(a) Let $x =$ the number of small photos and $y =$ the number of large photos.

To find the system of equations translate the information:

She wants to have at least 25 photos.

The number of small plus the number of large should be at least 25.

$$x + y \geq 25$$

\$4 for each small and \$10 for each large must be no more than \$200.

$$4x + 10 \leq 200$$

The number of small photos must be greater than or equal to 0.

$$x \geq 0$$

The number of large photos must be greater than or equal to 0.

$$y \geq 0$$

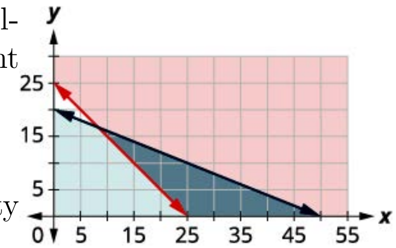
We have our system of inequalities:

$$\begin{cases} x + y \geq 25 \\ 4x + 10 \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

(b) Since $x \geq 0$ and $y \geq 0$, all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $x + y \geq 25$, graph $x + y = 25$ as a solid line.

Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.



To graph $4x + 10 \leq 200$, graph $4x + 10 = 200$ as a solid line.

Choose $(0, 0)$ as a test point. Since it does make the inequality true, shade (blue) the side that includes the point $(0, 0)$.

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly-shaded section are included in the solution as are the points on the x-axis from $(25, 0)$ to $(55, 0)$.

(c) To determine if 10 small and 20 large photos would work, we look at the graph to see if the point $(10, 20)$ is in the solution region. We could also test the point to see if it is a solution of both equations.

noi It is not, Christy would not display 10 small and 20 large photos.

(d) To determine if 20 small and 10 large photos would work, we look at the graph to see if the point (20, 10) is in the solution region. We could also test the point to see if it is a solution of both equations.

It is, so Christy could choose to display 20 small and 10 large photos.

Notice that we could also test the possible solutions by substituting the values into each inequality.

Try It #13: A trailer can carry a maximum weight of 160 pounds and a maximum volume of 15 cubic feet. A microwave oven weighs 30 pounds and has 2 cubic feet of volume, while a printer weighs 20 pounds and has 3 cubic feet of space.

(a) Write a system of inequalities to model this situation.

(b) Graph the system.

(c) Could 4 microwaves and 2 printers be carried on this trailer?

(d) Could 7 microwaves and 3 printers be carried on this trailer?

Try It #14: Mary needs to purchase supplies of answer sheets and pencils for a standardized test to be given to the juniors at her high school. The number of the answer sheets needed is at least 5 more than the number of pencils. The pencils cost \$2 and the answer sheets cost \$1. Mary's budget for these supplies allows for a maximum cost of \$400.

(a) Write a system of inequalities to model this situation.

(b) Graph the system.

(c) Could Mary purchase 100 pencils and 100 answer sheets?

(d) Could Mary purchase 150 pencils and 150 answer sheets?

When we use variables other than x and y to define an unknown quantity, we must change the names of the axes of the graph as well.

Example 8

Omar needs to eat at least 800 calories before going to his team practice. All he wants is hamburgers and cookies, and he doesn't want to spend more than \$5. At the hamburger restaurant near his college, each hamburger has 240 calories and costs \$1.40. Each cookie has 160 calories and costs \$0.50.

(a) Write a system of inequalities to model this situation.

(b) Graph the system.

(c) Could he eat 3 hamburgers and 1 cookie?

(d) Could he eat 2 hamburgers and 4 cookies?

Solution:

(a) Let $h =$ the number of hamburgers and $c =$ the number of cookies.

To find the system of equations translate the information:

The calories from hamburgers at 240 calories each, plus the calories from cookies at 160 calories each must be more that 800.

$$240h + 160c \geq 800$$

The amount spent on hamburgers, at \$1.40 each, plus the amount spent on cookies, at \$0.50 each, must be no more than \$5.00.

$$1.40h + 0.50c \leq 5$$

The number of hamburgers must be greater than or equal to 0.

$$h \geq 0$$

The number of cookies must be greater than or equal to 0.

$$c \geq 0$$

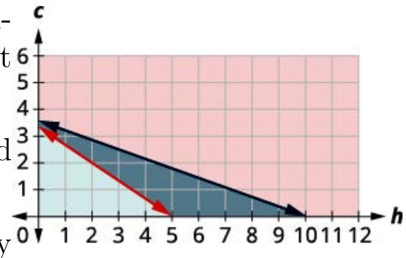
We have our system of inequalities:

$$\begin{cases} 240h + 160c \geq 800 \\ 1.40h + 0.50c \leq 5 \\ h \geq 0 \\ c \geq 0 \end{cases}$$

(b) Since $h \geq 0$ and $c \geq 0$, all solutions will be in the first quadrant. As a result, our graph shows only quadrant one.

To graph $240h + 160c \geq 800$, graph $240h + 160c = 800$ as a solid line.

Choose $(0, 0)$ as a test point. Since it does not make the inequality true, shade (red) the side that does not include the point $(0, 0)$.



To graph $1.40h + 0.50c \leq 5$, graph $1.40h + 0.50c = 5$ as a solid line.

Choose $(0, 0)$ as a test point. Since it does make the inequality true, shade (blue) the side that includes the point $(0, 0)$.

The solution of the system is the region of the graph that is shaded the darkest. The boundary line sections that border the darkly shaded section are included in the solution as are the points on the x-axis from $(5, 0)$ to $(10, 0)$.

(c) To determine if 3 hamburgers and 2 cookies would meet Omar's criteria, we see if the point (3, 2) is in the solution region.

It is, so Omar might choose to eat 3 hamburgers and 2 cookies.

(d) To determine if 2 hamburgers and 4 cookies would meet Omar's criteria, we see if the point (2, 4) is in the solution region.

It is, Omar might choose to eat 2 hamburgers and 4 cookies. We could also test the possible solutions by substituting the values into each inequality.

Try It #15: Tension needs to eat at least an extra 1,000 calories a day to prepare for running a marathon. He has only \$25 to spend on the extra food he needs and will spend it on \$0.75 donuts which have 360 calories each and \$2 energy drinks which have 110 calories.

(a) Write a system of inequalities to model this situation.

(b) Graph the system.

(c) Can he buy 8 donuts and 4 energy drinks and satisfy his caloric needs?

(d) Can he buy 1 donut and 3 energy drinks and satisfy his caloric needs?

Try It #16: Philips doctor tells him he should add at least 1,000 more calories per day to his usual diet. Philip wants to buy protein bars that cost \$1.80 each and have 140 calories and juice that costs \$1.25 per bottle and have 125 calories. He doesn't want to spend more than \$12.

(a) Write a system of inequalities to model this situation.

(b) Graph the system.

(c) Can he buy 3 protein bars and 5 bottles of juice?

(d) Can he buy 5 protein bars and 3 bottles of juice?

Determine Whether an Ordered Pair is a Solution of a System of Linear Inequalities

In the following exercises, determine whether each ordered pair is a solution to the system.

280. $\begin{cases} 3x + y > 5 \\ 2x - y \leq 10 \end{cases}$

ⓐ (3, −3)

ⓑ (7, 1)

281. $\begin{cases} 4x - y < 10 \\ -2x + 2y > -8 \end{cases}$

ⓐ (5, −2)

ⓑ (−1, 3)

282. $\begin{cases} y > \frac{2}{3}x - 5 \\ x + \frac{1}{2}y \leq 4 \end{cases}$

ⓐ (6, −4)

ⓑ (3, 0)

283. $\begin{cases} y < \frac{3}{2}x + 3 \\ \frac{3}{4}x - 2y < 5 \end{cases}$

ⓐ (−4, −1)

ⓑ (8, 3)

284. $\begin{cases} 7x + 2y > 14 \\ 5x - y \leq 8 \end{cases}$

ⓐ (2, 3)

ⓑ (7, −1)

285. $\begin{cases} 6x - 5y < 20 \\ -2x + 7y > -8 \end{cases}$

ⓐ (1, −3)

ⓑ (−4, 4)

Solve a System of Linear Inequalities by Graphing

In the following exercises, solve each system by graphing.

286. $\begin{cases} y \leq 3x + 2 \\ y > x - 1 \end{cases}$

287. $\begin{cases} y < -2x + 2 \\ y \geq -x - 1 \end{cases}$

288. $\begin{cases} y < 2x - 1 \\ y \leq -\frac{1}{2}x + 4 \end{cases}$

289. $\begin{cases} y \geq -\frac{2}{3}x + 2 \\ y > 2x - 3 \end{cases}$

290. $\begin{cases} x - y > 1 \\ y < -\frac{1}{4}x + 3 \end{cases}$

291. $\begin{cases} x + 2y < 4 \\ y < x - 2 \end{cases}$

292. $\begin{cases} 3x - y \geq 6 \\ y \geq -\frac{1}{2}x \end{cases}$

293. $\begin{cases} 2x + 4y \geq 8 \\ y \leq \frac{3}{4}x \end{cases}$

294. $\begin{cases} 2x - 5y < 10 \\ 3x + 4y \geq 12 \end{cases}$

295. $\begin{cases} 3x - 2y \leq 6 \\ -4x - 2y > 8 \end{cases}$

296. $\begin{cases} 2x + 2y > -4 \\ -x + 3y \geq 9 \end{cases}$

297. $\begin{cases} 2x + y > -6 \\ -x + 2y \geq -4 \end{cases}$

298. $\begin{cases} x - 2y < 3 \\ y \leq 1 \end{cases}$

299. $\begin{cases} x - 3y > 4 \\ y \leq -1 \end{cases}$

300. $\begin{cases} y \geq -\frac{1}{2}x - 3 \\ x \leq 2 \end{cases}$

301. $\begin{cases} y \leq -\frac{2}{3}x + 5 \\ x \geq 3 \end{cases}$

302. $\begin{cases} y \geq \frac{3}{4}x - 2 \\ y < 2 \end{cases}$

303. $\begin{cases} y \leq -\frac{1}{2}x + 3 \\ y < 1 \end{cases}$

304. $\begin{cases} 3x - 4y < 8 \\ x < 1 \end{cases}$

305. $\begin{cases} -3x + 5y > 10 \\ x > -1 \end{cases}$

306. $\begin{cases} x \geq 3 \\ y \leq 2 \end{cases}$

307. $\begin{cases} x \leq -1 \\ y \geq 3 \end{cases}$

308. $\begin{cases} 2x + 4y > 4 \\ y \leq -\frac{1}{2}x - 2 \end{cases}$

309. $\begin{cases} x - 3y \geq 6 \\ y > \frac{1}{3}x + 1 \end{cases}$

310. $\begin{cases} -2x + 6y < 0 \\ 6y > 2x + 4 \end{cases}$

311. $\begin{cases} -3x + 6y > 12 \\ 4y \leq 2x - 4 \end{cases}$

312. $\begin{cases} y \geq -3x + 2 \\ 3x + y > 5 \end{cases}$

313. $\begin{cases} y \geq \frac{1}{2}x - 1 \\ -2x + 4y \geq 4 \end{cases}$

314. $\begin{cases} y \leq -\frac{1}{4}x - 2 \\ x + 4y < 6 \end{cases}$

315. $\begin{cases} y \geq 3x - 1 \\ -3x + y > -4 \end{cases}$

316. $\begin{cases} 3y > x + 2 \\ -2x + 6y > 8 \end{cases}$

317. $\begin{cases} y < \frac{3}{4}x - 2 \\ -3x + 4y < 7 \end{cases}$

Solve Applications of Systems of Inequalities

In the following exercises, translate to a system of inequalities and solve.

318. Caitlyn sells her drawings at the county fair. She wants to sell at least 60 drawings and has portraits and landscapes. She sells the portraits for $15 and the landscapes for $10. She needs to sell at least $800 worth of drawings in order to earn a profit.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Will she make a profit if she sells 20 portraits and 35 landscapes?

ⓓ Will she make a profit if she sells 50 portraits and 20 landscapes?

319. Jake does not want to spend more than $50 on bags of fertilizer and peat moss for his garden. Fertilizer costs $2 a bag and peat moss costs $5 a bag. Jake's van can hold at most 20 bags.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can he buy 15 bags of fertilizer and 4 bags of peat moss?

ⓓ Can he buy 10 bags of fertilizer and 10 bags of peat moss?

320. Reiko needs to mail her Christmas cards and packages and wants to keep her mailing costs to no more than $500. The number of cards is at least 4 more than twice the number of packages. The cost of mailing a card (with pictures enclosed) is $3 and for a package the cost is $7.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can she mail 60 cards and 26 packages?

ⓓ Can she mail 90 cards and 40 packages?

321. Juan is studying for his final exams in chemistry and algebra. he knows he only has 24 hours to study, and it will take him at least three times as long to study for algebra than chemistry.

ⓐ Write a system of inequalities to model this situation.

ⓑ Graph the system.

ⓒ Can he spend 4 hours on chemistry and 20 hours on algebra?

ⓓ Can he spend 6 hours on chemistry and 18 hours on algebra?

CHAPTER 2 REVIEW

Key Terms

absolute value equation an equation in which the variable appears in absolute value bars, typically with two solutions, one accounting for the positive expression and one for the negative expression

area in square units, the area formula used in this section is used to find the area of any two-dimensional rectangular region: $A = LW$

Cartesian coordinate system a grid system designed with perpendicular axes invented by René Descartes

completing the square a process for solving quadratic equations in which terms are added to or subtracted from both sides of the equation in order to make one side a perfect square

complex conjugate a complex number containing the same terms as another complex number, but with the opposite operator. Multiplying a complex number by its conjugate yields a real number.

complex number the sum of a real number and an imaginary number; the standard form is $a + bi$, where a is the real part and b is the complex part.

complex plane the coordinate plane in which the horizontal axis represents the real component of a complex number, and the vertical axis represents the imaginary component, labeled i.

compound inequality a problem or a statement that includes two inequalities

conditional equation an equation that is true for some values of the variable

discriminant the expression under the radical in the quadratic formula that indicates the nature of the solutions, real or complex, rational or irrational, single or double roots.

distance formula a formula that can be used to find the length of a line segment if the endpoints are known

equation in two variables a mathematical statement, typically written in x and y, in which two expressions are equal

equations in quadratic form equations with a power other than 2 but with a middle term with an exponent that is one-half the exponent of the leading term

extraneous solutions any solutions obtained that are not valid in the original equation

graph in two variables the graph of an equation in two variables, which is always shown in two variables in the two-dimensional plane

identity equation an equation that is true for all values of the variable

imaginary number the square root of -1: $i = \sqrt{-1}$.

inconsistent equation an equation producing a false result

intercepts the points at which the graph of an equation crosses the x-axis and the y-axis

interval an interval describes a set of numbers within which a solution falls

interval notation a mathematical statement that describes a solution set and uses parentheses or brackets to indicate where an interval begins and ends

linear equation an algebraic equation in which each term is either a constant or the product of a constant and the first power of a variable

linear inequality similar to a linear equation except that the solutions will include sets of numbers

midpoint formula a formula to find the point that divides a line segment into two parts of equal length

ordered pair a pair of numbers indicating horizontal displacement and vertical displacement from the origin; also known as a coordinate pair, (x, y)

origin the point where the two axes cross in the center of the plane, described by the ordered pair (0, 0)

perimeter in linear units, the perimeter formula is used to find the linear measurement, or outside length and width, around a two-dimensional regular object; for a rectangle: $P = 2L + 2W$

polynomial equation an equation containing a string of terms including numerical coefficients and variables raised to whole-number exponents

Pythagorean Theorem a theorem that states the relationship among the lengths of the sides of a right triangle, used to solve right triangle problems

quadrant one quarter of the coordinate plane, created when the axes divide the plane into four sections

quadratic equation an equation containing a second-degree polynomial; can be solved using multiple methods

quadratic formula a formula that will solve all quadratic equations

radical equation an equation containing at least one radical term where the variable is part of the radicand

rational equation an equation consisting of a fraction of polynomials

slope the change in y-values over the change in x-values

solution set the set of all solutions to an equation

square root property one of the methods used to solve a quadratic equation, in which the x^2 term is isolated so that the square root of both sides of the equation can be taken to solve for x

volume in cubic units, the volume measurement includes length, width, and depth: $V = LWH$

***x*-axis** the common name of the horizontal axis on a coordinate plane; a number line increasing from left to right

***x*-coordinate** the first coordinate of an ordered pair, representing the horizontal displacement and direction from the origin

***x*-intercept** the point where a graph intersects the x-axis; an ordered pair with a y-coordinate of zero

***y*-axis** the common name of the vertical axis on a coordinate plane; a number line increasing from bottom to top

***y*-coordinate** the second coordinate of an ordered pair, representing the vertical displacement and direction from the origin

***y*-intercept** a point where a graph intercepts the y-axis; an ordered pair with an x-coordinate of zero

zero-product property the property that formally states that multiplication by zero is zero, so that each factor of a quadratic equation can be set equal to zero to solve equations

Key Equations

quadratic formula $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Key Concepts

2.1 The Rectangular Coordinate Systems and Graphs

- We can locate, or plot, points in the Cartesian coordinate system using ordered pairs, which are defined as displacement from the x-axis and displacement from the y-axis. See **Example 1**.
- An equation can be graphed in the plane by creating a table of values and plotting points. See **Example 2**.
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form $y =$ ______. See **Example 3**.
- Finding the x- and y-intercepts can define the graph of a line. These are the points where the graph crosses the axes. See **Example 4**.
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment. See **Example 5** and **Example 6**.
- The midpoint formula provides a method of finding the coordinates of the midpoint dividing the sum of the x-coordinates and the sum of the y-coordinates of the endpoints by 2. See **Example 7** and **Example 8**.

2.2 Linear Equations in One Variable

- We can solve linear equations in one variable in the form $ax + b = 0$ using standard algebraic properties. See **Example 1** and **Example 2**.
- A rational expression is a quotient of two polynomials. We use the LCD to clear the fractions from an equation. See **Example 3** and **Example 4**.
- All solutions to a rational equation should be verified within the original equation to avoid an undefined term, or zero in the denominator. See **Example 5, Example 6,** and **Example 7**.
- Given two points, we can find the slope of a line using the slope formula. See **Example 8**.

- We can identify the slope and y-intercept of an equation in slope-intercept form. See **Example 9**.
- We can find the equation of a line given the slope and a point. See **Example 10**.
- We can also find the equation of a line given two points. Find the slope and use the point-slope formula. See **Example 11**.
- The standard form of a line has no fractions. See **Example 12**.
- Horizontal lines have a slope of zero and are defined as $y = c$, where c is a constant.
- Vertical lines have an undefined slope (zero in the denominator), and are defined as $x = c$, where c is a constant. See **Example 13**.
- Parallel lines have the same slope and different y-intercepts. See **Example 14** and **Example 15**.
- Perpendicular lines have slopes that are negative reciprocals of each other unless one is horizontal and the other is vertical. See **Example 16**.

2.3 Models and Applications

- A linear equation can be used to solve for an unknown in a number problem. See **Example 1**.
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities. See **Example 2**.
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d = rt$ formula. See **Example 3**.
- Many geometry problems are solved using the perimeter formula $P = 2L + 2W$, the area formula $A = LW$, or the volume formula $V = LWH$. See **Example 4**, **Example 5**, and **Example 6**.

2.4 Complex Numbers

- The square root of any negative number can be written as a multiple of i. See **Example 1**.
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis. See **Example 2**.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts. See **Example 3**.
- Complex numbers can be multiplied and divided.
  - To multiply complex numbers, distribute just as with polynomials. See **Example 4** and **Example 5**.
  - To divide complex numbers, multiply both numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator. See **Example 6** and **Example 7**.
- The powers of i are cyclic, repeating every fourth one. See **Example 8**.

2.5 Quadratic Equations

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of 1 or if the equation is a difference of squares. The zero-product property is then used to find solutions. See **Example 1**, **Example 2**, and **Example 3**.
- Many quadratic equations with a leading coefficient other than 1 can be solved by factoring using the grouping method. See **Example 4** and **Example 5**.
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution. See **Example 6** and **Example 7**.
- Completing the square is a method of solving quadratic equations when the equation cannot be factored. See **Example 8**.
- A highly dependable method for solving quadratic equations is the quadratic formula, based on the coefficients and the constant term in the equation. See **Example 9** and **Example 10**.
- The discriminant is used to indicate the nature of the roots that the quadratic equation will yield: real or complex, rational or irrational, and how many of each. See **Example 11**.

- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation. See **Example 12**.

2.6 Other Types of Equations

- Rational exponents can be rewritten several ways depending on what is most convenient for the problem. To solve, both sides of the equation are raised to a power that will render the exponent on the variable equal to 1. See **Example 1**, **Example 2**, and **Example 3**.
- Factoring extends to higher-order polynomials when it involves factoring out the GCF or factoring by grouping. See **Example 4** and **Example 5**.
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index. See **Example 6** and **Example 7**.
- To solve absolute value equations, we need to write two equations, one for the positive value and one for the negative value. See **Example 8**.
- Equations in quadratic form are easy to spot, as the exponent on the first term is double the exponent on the second term and the third term is a constant. We may also see a binomial in place of the single variable. We use substitution to solve. See **Example 9** and **Example 10**.
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form. See **Example 11**.

2.7 Linear Inequalities and Absolute Value Inequalities

- Interval notation is a method to indicate the solution set to an inequality. Highly applicable in calculus, it is a system of parentheses and brackets that indicate what numbers are included in a set and whether the endpoints are included as well. See **Table 1** and **Example 1** and **Example 2**.
- Solving inequalities is similar to solving equations. The same algebraic rules apply, except for one: multiplying or dividing by a negative number reverses the inequality. See **Example 3**, **Example 4**, **Example 5**, and **Example 6**.
- Compound inequalities often have three parts and can be rewritten as two independent inequalities. Solutions are given by boundary values, which are indicated as a beginning boundary or an ending boundary in the solutions to the two inequalities. See **Example 7** and **Example 8**.
- Absolute value inequalities will produce two solution sets due to the nature of absolute value. We solve by writing two equations: one equal to a positive value and one equal to a negative value. See **Example 9** and **Example 10**.
- Absolute value inequalities can also be solved by graphing. At least we can check the algebraic solutions by graphing, as we cannot depend on a visual for a precise solution. See **Example 11**.

CHAPTER 2 REVIEW EXERCISES

THE RECTANGULAR COORDINATE SYSTEMS AND GRAPHS

For the following exercises, find the x-intercept and the y-intercept without graphing.

1. $4x - 3y = 12$

2. $2y - 4 = 3x$

For the following exercises, solve for y in terms of x, putting the equation in slope–intercept form.

3. $5x = 3y - 12$

4. $2x - 5y = 7$

For the following exercises, find the distance between the two points.

5. $(-2, 5)(4, -1)$

6. $(-12, -3)(-1, 5)$

7. Find the distance between the two points $(-71{,}432)$ and $(511{,}218)$ using your calculator, and round your answer to the nearest thousandth.

For the following exercises, find the coordinates of the midpoint of the line segment that joins the two given points.

8. $(-1, 5)$ and $(4, 6)$

9. $(-13, 5)$ and $(17, 18)$

For the following exercises, construct a table and graph the equation by plotting at least three points.

10. $y = \frac{1}{2}x + 4$

11. $4x - 3y = 6$

LINEAR EQUATIONS IN ONE VARIABLE

For the following exercises, solve for x.

12. $5x + 2 = 7x - 8$

13. $3(x + 2) - 10 = x + 4$

14. $7x - 3 = 5$

15. $12 - 5(x + 1) = 2x - 5$

16. $\frac{2x}{3} - \frac{3}{4} = \frac{x}{6} + \frac{21}{4}$

For the following exercises, solve for x. State all x-values that are excluded from the solution set.

17. $\frac{x}{x^2 - 9} + \frac{4}{x + 3} = \frac{3}{x^2 - 9}$ $x \neq 3, -3$

18. $\frac{1}{2} + \frac{2}{x} = \frac{3}{4}$

For the following exercises, find the equation of the line using the point-slope formula.

19. Passes through these two points: $(-2, 1),(4, 2)$.

20. Passes through the point $(-3, 4)$ and has a slope of $-\frac{1}{3}$.

21. Passes through the point $(-3, 4)$ and is parallel to the graph $y = \frac{2}{3}x + 5$.

22. Passes through these two points: $(5, 1),(5, 7)$.

MODELS AND APPLICATIONS

For the following exercises, write and solve an equation to answer each question.

23. The number of males in the classroom is five more than three times the number of females. If the total number of students is 73, how many of each gender are in the class?

24. A man has 72 ft of fencing to put around a rectangular garden. If the length is 3 times the width, find the dimensions of his garden.

25. A truck rental is \$25 plus \$.30/mi. Find out how many miles Ken traveled if his bill was \$50.20.

COMPLEX NUMBERS

For the following exercises, use the quadratic equation to solve.

26. $x^2 - 5x + 9 = 0$

27. $2x^2 + 3x + 7 = 0$

For the following exercises, name the horizontal component and the vertical component.

28. $4 - 3i$

29. $-2 - i$

For the following exercises, perform the operations indicated.

30. $(9 - i) - (4 - 7i)$

31. $(2 + 3i) - (-5 - 8i)$

32. $2\sqrt{-75} + 3\sqrt{25}$

33. $\sqrt{-16} + 4\sqrt{-9}$

34. $-6i(i - 5)$

35. $(3 - 5i)^2$

36. $\sqrt{-4} \cdot \sqrt{-12}$

37. $\sqrt{-2}\left(\sqrt{-8} - \sqrt{5}\right)$

38. $\dfrac{2}{5 - 3i}$

39. $\dfrac{3 + 7i}{i}$

QUADRATIC EQUATIONS

For the following exercises, solve the quadratic equation by factoring.

40. $2x^2 - 7x - 4 = 0$

41. $3x^2 + 18x + 15 = 0$

42. $25x^2 - 9 = 0$

43. $7x^2 - 9x = 0$

For the following exercises, solve the quadratic equation by using the square-root property.

44. $x^2 = 49$

45. $(x - 4)^2 = 36$

For the following exercises, solve the quadratic equation by completing the square.

46. $x^2 + 8x - 5 = 0$

47. $4x^2 + 2x - 1 = 0$

For the following exercises, solve the quadratic equation by using the quadratic formula. If the solutions are not real, state *No real solution*.

48. $2x^2 - 5x + 1 = 0$

49. $15x^2 - x - 2 = 0$

For the following exercises, solve the quadratic equation by the method of your choice.

50. $(x - 2)^2 = 16$

51. $x^2 = 10x + 3$

OTHER TYPES OF EQUATIONS

For the following exercises, solve the equations.

52. $x^{\frac{3}{2}} = 27$

53. $x^{\frac{1}{2}} - 4x^{\frac{1}{4}} = 0$

54. $4x^3 + 8x^2 - 9x - 18 = 0$

55. $3x^5 - 6x^3 = 0$

56. $\sqrt{x+9} = x - 3$

57. $\sqrt{3x+7} + \sqrt{x+2} = 1$

58. $|3x - 7| = 5$

59. $|2x + 3| - 5 = 9$

LINEAR INEQUALITIES AND ABSOLUTE VALUE INEQUALITIES

For the following exercises, solve the inequality. Write your final answer in interval notation.

60. $5x - 8 \leq 12$

61. $-2x + 5 > x - 7$

62. $\frac{x-1}{3} + \frac{x+2}{5} \leq \frac{3}{5}$

63. $|3x + 2| + 1 \leq 9$

64. $|5x - 1| > 14$

65. $|x - 3| < -4$

For the following exercises, solve the compound inequality. Write your answer in interval notation.

66. $-4 < 3x + 2 \leq 18$

67. $3y < 1 - 2y < 5 + y$

For the following exercises, graph as described.

68. Graph the absolute value function and graph the constant function. Observe the points of intersection and shade the x-axis representing the solution set to the inequality. Show your graph and write your final answer in interval notation.
$|x + 3| \geq 5$

69. Graph both straight lines (left-hand side being y_1 and right-hand side being y_2) on the same axes. Find the point of intersection and solve the inequality by observing where it is true comparing the y-values of the lines. See the interval where the inequality is true.
$x + 3 < 3x - 4$

CHAPTER 2 PRACTICE TEST

1. Graph the following: $2y = 3x + 4$.

2. Find the x- and y-intercepts for the following: $2x - 5y = 6$.

3. Find the x- and y-intercepts of this equation, and sketch the graph of the line using just the intercepts plotted. $3x - 4y = 12$

4. Find the exact distance between $(5, -3)$ and $(-2, 8)$. Find the coordinates of the midpoint of the line segment joining the two points.

5. Write the interval notation for the set of numbers represented by $\{x | x \le 9\}$.

6. Solve for x: $5x + 8 = 3x - 10$.

7. Solve for x: $3(2x - 5) - 3(x - 7) = 2x - 9$.

8. Solve for x: $\frac{x}{2} + 1 = \frac{4}{x}$

9. Solve for x: $\frac{5}{x+4} = 4 + \frac{3}{x-2}$.

10. The perimeter of a triangle is 30 in. The longest side is 2 less than 3 times the shortest side and the other side is 2 more than twice the shortest side. Find the length of each side.

11. Solve for x. Write the answer in simplest radical form. $\frac{x^2}{3} - x = -\frac{1}{2}$

12. Solve: $3x - 8 \le 4$.

13. Solve: $|2x + 3| < 5$.

14. Solve: $|3x - 2| \ge 4$.

For the following exercises, find the equation of the line with the given information.

15. Passes through the points $(-4, 2)$ and $(5, -3)$.

16. Has an undefined slope and passes through the point $(4, 3)$.

17. Passes through the point $(2, 1)$ and is perpendicular to $y = -\frac{2}{5}x + 3$.

18. Add these complex numbers: $(3 - 2i) + (4 - i)$.

19. Simplify: $\sqrt{-4} + 3\sqrt{-16}$.

20. Multiply: $5i(5 - 3i)$.

21. Divide: $\frac{4-i}{2+3i}$.

22. Solve this quadratic equation and write the two complex roots in $a + bi$ form: $x^2 - 4x + 7 = 0$.

23. Solve: $(3x - 1)^2 - 1 = 24$.

24. Solve: $x^2 - 6x = 13$.

25. Solve: $4x^2 - 4x - 1 = 0$

26. Solve: $\sqrt{x - 7} = x - 7$

27. Solve: $2 + \sqrt{12 - 2x} = x$

28. Solve: $(x - 1)^{\frac{2}{3}} = 9$

For the following exercises, find the real solutions of each equation by factoring.

29. $2x^3 - x^2 - 8x + 4 = 0$

30. $(x + 5)^2 - 3(x + 5) - 4 = 0$

LEARNING OBJECTIVES

In this section, you will:

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

3.1 FUNCTIONS AND FUNCTION NOTATION

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set consisting of the first components of each ordered pair is called the **domain** and the set consisting of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)\}$$

The domain is $\{1, 2, 3, 4, 5\}$. The range is $\{2, 4, 6, 8, 10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an **output** value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single element in the range to each element in the domain. In other words, no x-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, $\{1, 2, 3, 4, 5\}$, is paired with exactly one element in the range, $\{2, 4, 6, 8, 10\}$.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd}, 1), (\text{even}, 2), (\text{odd}, 3), (\text{even}, 4), (\text{odd}, 5)\}$$

Notice that each element in the domain, $\{\text{even}, \text{odd}\}$ is *not* paired with exactly one element in the range, $\{1, 2, 3, 4, 5\}$. For example, the term "odd" corresponds to three values from the domain, $\{1, 3, 5\}$ and the term "even" corresponds to two values from the range, $\{2, 4\}$. This violates the definition of a function, so this relation is not a function. **Figure 1** compares relations that are functions and not functions.

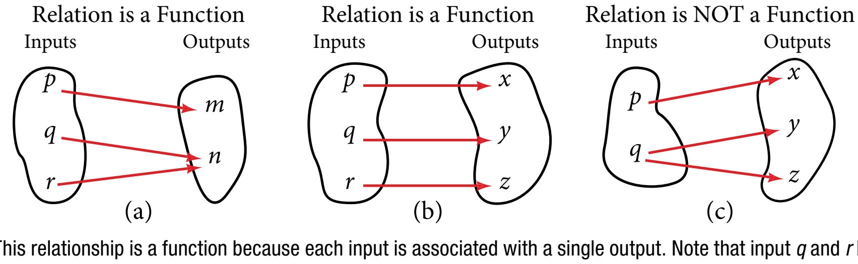


Figure 1 (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n.
(b) This relationship is also a function. In this case, each input is associated with a single output.
(c) This relationship is not a function because input q is associated with two different outputs.

> ***function***
> A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."
>
> The **input** values make up the **domain**, and the **output** values make up the **range**.

How To...

Given a relationship between two quantities, determine whether the relationship is a function.

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 1 Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in **Figure 2** consists of items and their prices.

a. Is price a function of the item? **b.** Is the item a function of the price?

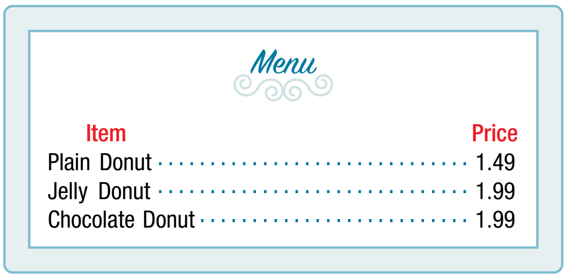


Figure 2

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See **Figure 2**.

Each item on the menu has only one price, so the price is a function of the item.

b. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See **Figure 3**.

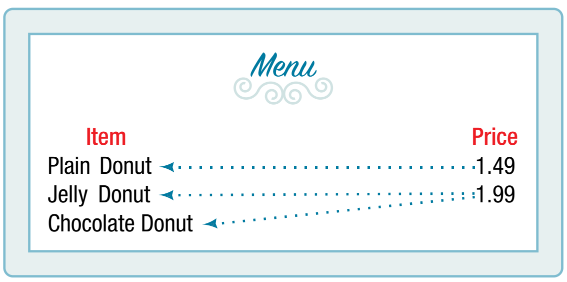


Figure 3

Therefore, the item is a not a function of price.

Example 2 Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade-point average. Is grade-point average a function of the percent grade? Is the percent grade a function of the grade-point average? **Table 1** shows a possible rule for assigning grade points.

Percent grade	0-56	57-61	62-66	67-71	72-77	78-86	87-91	92-100
Grade-point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Table 1

Solution For any percent grade earned, there is an associated grade-point average, so the grade-point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade-point average.

In the grading system given, there is a range of percent grades that correspond to the same grade-point average. For example, students who receive a grade-point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade-point average

Try It #1

Table 2[1] lists the five greatest baseball players of all time in order of rank.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

Table 2

a. Is the rank a function of the player name?

b. Is the player name a function of the rank?

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into graphing calculators and computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g, and h are often used to represent functions just as we use x, y, and z to represent numbers and A, B, and C to represent sets.

h is f of a	We name the function f; height is a function of age.
$h = f(a)$	We use parentheses to indicate the function input.
$f(a)$	We name the function f; the expression is read as "f of a."

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a + b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

function notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

1 http://www.baseball-almanac.com/legendary/lisn100.shtml. Accessed 3/24/2014.

Example 3 Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution The number of days in a month is a function of the name of the month, so if we name the function f, we write days $= f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

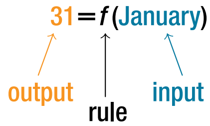


Figure 4

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis *Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.*

Example 4 Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Try It #2

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Q & A...

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 3 lists the input number of each month (January = 1, February = 2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 3

Table 4 defines a function $Q = g(n)$. Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 4

Table 5 below displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

Table 5

How To...

Given a table of input and output values, determine whether the table represents a function.

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example 5 Identifying Tables that Represent Functions

Which table, **Table 6**, **Table 7**, or **Table 8**, represents a function (if any)?

Input	Output
2	1
5	3
8	6

Table 6

Input	Output
−3	5
0	1
4	5

Table 7

Input	Output
1	0
5	2
5	4

Table 8

Solution **Table 6** and **Table 7** define functions. In both, each input value corresponds to exactly one output value. **Table 8** does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by **Table 6** can be represented by writing

$$f(2) = 1, f(5) = 3, \text{and } f(8) = 6$$

Similarly, the statements

$$g(-3) = 5, g(0) = 1, \text{and } g(4) = 5$$

represent the function in table **Table 7.**

Table 8 cannot be expressed in a similar way because it does not represent a function.

Try It #3

Does **Table 9** represent a function?

Input	Output
1	10
2	100
3	1000

Table 9

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To...

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 6 **Evaluating Functions at Specific Values**

Evaluate $f(x) = x^2 + 3x - 4$ at:

a. 2 **b.** a **c.** $a + h$ **d.** $\dfrac{f(a+h) - f(a)}{h}$

Solution Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{2ah + h^2 + 3h}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out } h. \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

Example 7 Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$h(p) = p^2 + 2p$$
$$h(4) = (4)^2 + 2\,(4)$$
$$= 16 + 8$$
$$= 24$$

Therefore, for an input of 4, we have an output of 24.

Try It #4

Given the function $g(m) = \sqrt{m - 4}$. Evaluate $g(5)$.

Example 8 Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

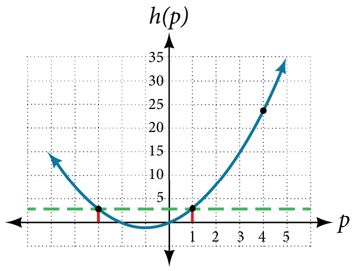
Solution

$$h(p) = 3$$
$$p^2 + 2p = 3 \qquad \text{Substitute the original function } h(p) = p^2 + 2p.$$
$$p^2 + 2p - 3 = 0 \qquad \text{Subtract 3 from each side.}$$
$$(p + 3)(p - 1) = 0 \qquad \text{Factor.}$$

If $(p + 3)(p - 1) = 0$, either $(p + 3) = 0$ or $(p - 1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p + 3) = 0, \quad p = -3$$
$$(p - 1) = 0, \quad p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in **Figure 5**. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	–3	–2	0	1	4
$h(p)$	3	0	0	3	24

Figure 5

Try It #5

Given the function $g(m) = \sqrt{m - 4}$, solve $g(m) = 2$.

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n + 6p = 12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How To…

Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves *only* the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 9 Finding an Equation of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Solution To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p =$ [expression involving n].

$$2n + 6p = 12$$

$$6p = 12 - 2n \qquad \text{Subtract } 2n \text{ from both sides.}$$

$$p = \frac{12 - 2n}{6} \qquad \text{Divide both sides by 6 and simplify.}$$

$$p = \frac{12}{6} - \frac{2n}{6}$$

$$p = 2 - \frac{1}{3}n$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Example 10 Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution First we subtract x^2 from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.

$$y = \pm\sqrt{1 - x^2}$$

$$= +\sqrt{1 - x^2} \quad \text{and} \quad -\sqrt{1 - x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$. If we graph both functions on a graphing calculator, we will get the upper and lower semicircles.

Try It #6

If $x - 8y^3 = 0$, express y as a function of x.

Q & A…

Are there relationships expressed by an equation that do represent a function but that still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See **Table 10**.[2]

Pet	Memory span in hours
Puppy	0.008
Adult dog	0.083
Cat	16
Goldfish	2160
Beta fish	3600

Table 10

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(\text{goldfish}) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To...

Given a function represented by a table, identify specific output and input values.

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 11 Evaluating and Solving a Tabular Function

Using **Table 11**,

a. Evaluate $g(3)$ **b.** Solve $g(n) = 6$.

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Table 11

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.

b. Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. **Table 11** shows two solutions: 2 and 4. When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

2 http://www.kgbanswers.com/how-long-is-a-dogs-memory-span/4221590. Accessed 3/24/2014.

Try It #7

Using **Table 11**, evaluate $g(1)$.

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 12 **Reading Function Values from a Graph**

Given the graph in **Figure 6**,

a. Evaluate $f(2)$. **b.** Solve $f(x) = 4$.

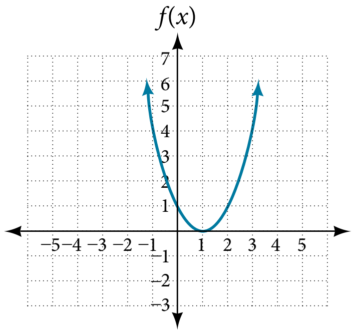


Figure 6

Solution

a. To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See **Figure 7**.

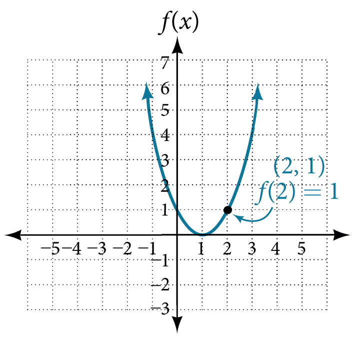


Figure 7

b. To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: -1 or 3. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4. See **Figure 8**.

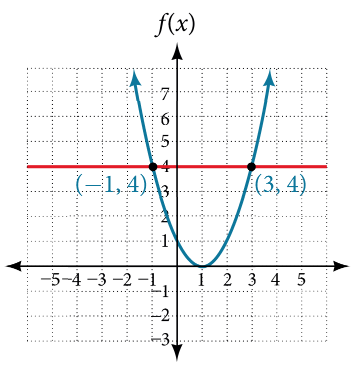


Figure 8

Try It #8

Using **Figure 7**, solve $f(x) = 1$.

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in **Figure 1** at the beginning of this chapter, the stock price was $1,000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1,000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in **Table 12**.

Letter grade	Grade-point average
A	4.0
B	3.0
C	2.0
D	1.0

Table 12

This grading system represents a one-to-one function, because each letter input yields one particular grade-point average output and each grade-point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in **Figure 1(a)** and **Figure 1(b)**. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

one-to-one function

A **one-to-one function** is a function in which each output value corresponds to exactly one input value. There are no repeated x- or y-values.

Example 13 Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius r.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $r = \sqrt{\frac{A}{\pi}}$ So the area of a circle is a one-to-one function of the circle's radius.

Try It #9

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Try It #10

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?

b. If so, is the function one-to-one?

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output value y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function consists of only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in **Figure 9** tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes (0, 2) and (6, 1) because the curve passes through those points.

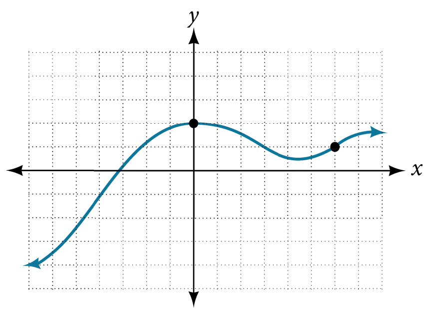


Figure 9

The **vertical line test** can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See **Figure 10**.

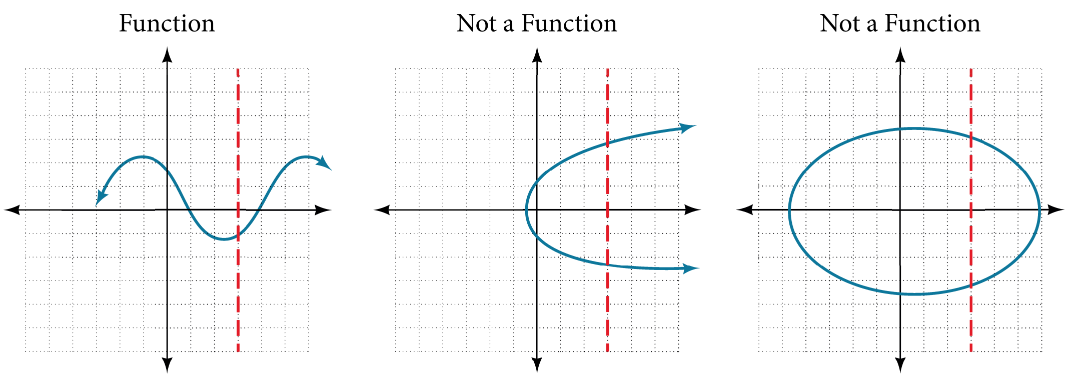


Figure 10

How To...

Given a graph, use the vertical line test to determine if the graph represents a function.

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 14 Applying the Vertical Line Test

Which of the graphs in **Figure 11** represent(s) a function $y = f(x)$?

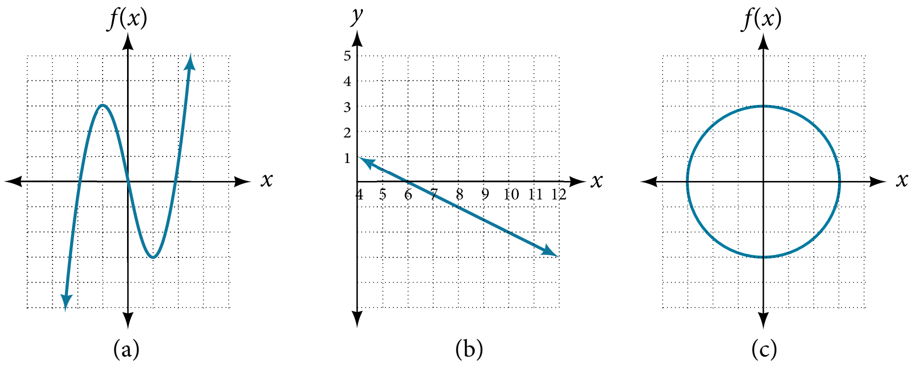


Figure 11

Solution If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of **Figure 11**. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in **Figure 12**.

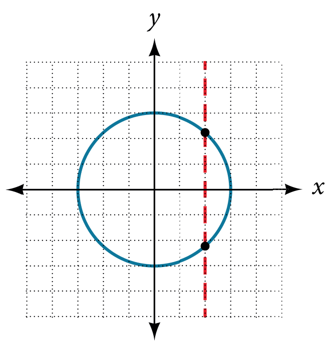


Figure 12

Try It #11

Does the graph in **Figure 13** represent a function?

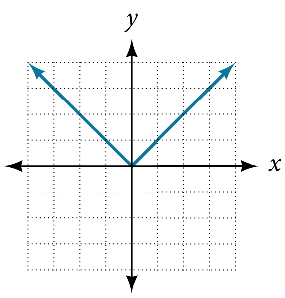


Figure 13

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

How To…

Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 15 Applying the Horizontal Line Test

Consider the functions shown in **Figure 11(a)** and **Figure 11(b)**. Are either of the functions one-to-one?

Solution The function in **Figure 11(a)** is not one-to-one. The horizontal line shown in **Figure 14** intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

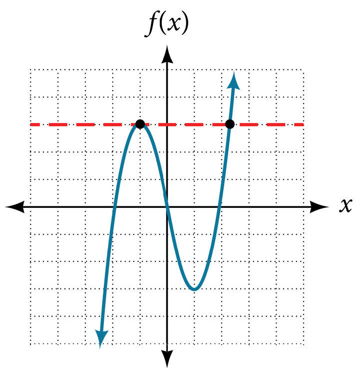
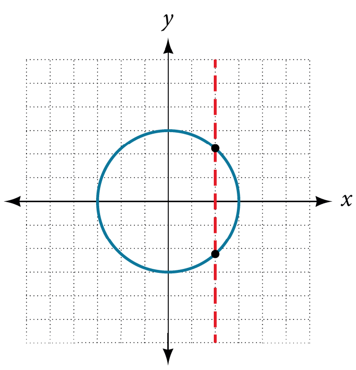


Figure 14

The function in **Figure 11(b)** is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Try It #12

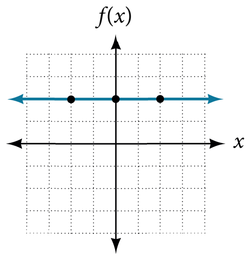
Is the graph shown here one-to-one?

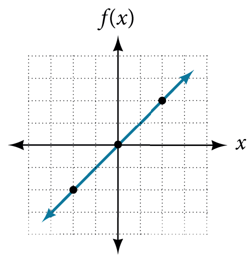


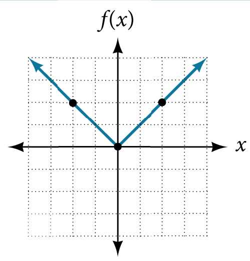
Identifying Basic Toolkit Functions

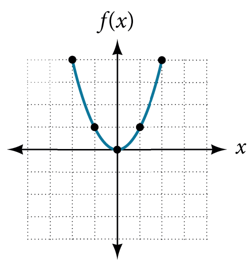
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our “toolkit functions,” which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

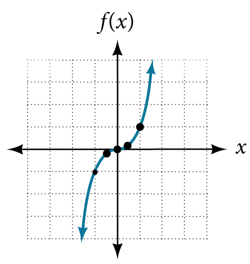
We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in **Table 13**.

<table>
<tr><th colspan="3">Toolkit Functions</th></tr>
<tr><th>Name</th><th>Function</th><th>Graph</th></tr>
<tr><td>Constant</td><td>$f(x) = c$, where c is a constant</td><td>

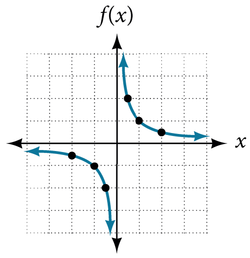
<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Identity</td><td>$f(x) = x$</td><td>

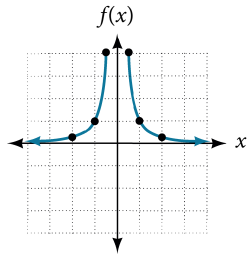
<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>–2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Absolute value</td><td>$f(x) = \lvert x \rvert$</td><td>

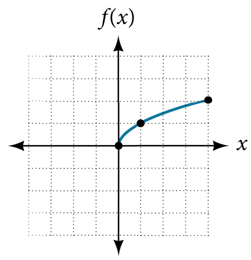
<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table></td></tr>
<tr><td>Quadratic</td><td>$f(x) = x^2$</td><td>

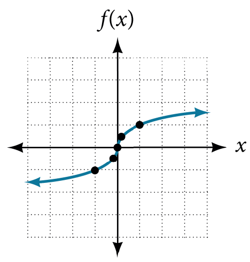
<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–2</td><td>4</td></tr><tr><td>–1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td></tr>
<tr><td>Cubic</td><td>$f(x) = x^3$</td><td>

<table><tr><th>x</th><th>$f(x)$</th></tr><tr><td>–1</td><td>–1</td></tr><tr><td>–0.5</td><td>–0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table></td></tr>
</table>

<table>
<tr><td>Reciprocal</td><td>$f(x) = \frac{1}{x}$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−0.5</td></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table></td></tr>
<tr><td>Reciprocal squared</td><td>$f(x) = \frac{1}{x^2}$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>0.25</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>−0.5</td><td>4</td></tr><tr><td>0.5</td><td>4</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.25</td></tr></table></td></tr>
<tr><td>Square root</td><td>$f(x) = \sqrt{x}$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table></td></tr>
<tr><td>Cube root</td><td>$f(x) = \sqrt[3]{x}$</td><td>

<table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table></td></tr>
</table>

Table 13

Access the following online resources for additional instruction and practice with functions.

- Determine if a Relation is a Function (http://openstaxcollege.org/l/relationfunction)
- Vertical Line Test (http://openstaxcollege.org/l/vertlinetest)
- Introduction to Functions (http://openstaxcollege.org/l/introtofunction)
- Vertical Line Test of Graph (http://openstaxcollege.org/l/vertlinegraph)
- One-to-one Functions (http://openstaxcollege.org/l/onetoone)
- Graphs as One-to-one Functions (http://openstaxcollege.org/l/graphonetoone)

3.1 SECTION EXERCISES

VERBAL

1. What is the difference between a relation and a function?

2. What is the difference between the input and the output of a function?

3. Why does the vertical line test tell us whether the graph of a relation represents a function?

4. How can you determine if a relation is a one-to-one function?

5. Why does the horizontal line test tell us whether the graph of a function is one-to-one?

ALGEBRAIC

For the following exercises, determine whether the relation represents a function.

6. $\{(a, b), (c, d), (a, c)\}$

7. $\{(a, b),(b, c),(c, c)\}$

For the following exercises, determine whether the relation represents y as a function of x.

8. $5x + 2y = 10$

9. $y = x^2$

10. $x = y^2$

11. $3x^2 + y = 14$

12. $2x + y^2 = 6$

13. $y = -2x^2 + 40x$

14. $y = \dfrac{1}{x}$

15. $x = \dfrac{3y + 5}{7y - 1}$

16. $x = \sqrt{1 - y^2}$

17. $y = \dfrac{3x + 5}{7x - 1}$

18. $x^2 + y^2 = 9$

19. $2xy = 1$

20. $x = y^3$

21. $y = x^3$

22. $y = \sqrt{1 - x^2}$

23. $x = \pm\sqrt{1 - y}$

24. $y = \pm\sqrt{1 - x}$

25. $y^2 = x^2$

26. $y^3 = x^2$

For the following exercises, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a + h)$.

27. $f(x) = 2x - 5$

28. $f(x) = -5x^2 + 2x - 1$

29. $f(x) = \sqrt{2 - x} + 5$

30. $f(x) = \dfrac{6x - 1}{5x + 2}$

31. $f(x) = |x - 1| - |x + 1|$

32. Given the function $g(x) = 5 - x^2$, simplify $\dfrac{g(x + h) - g(x)}{h}$, $h \neq 0$

33. Given the function $g(x) = x^2 + 2x$, simplify $\dfrac{g(x) - g(a)}{x - a}$, $x \neq a$

34. Given the function $k(t) = 2t - 1$:
- **a.** Evaluate $k(2)$.
- **b.** Solve $k(t) = 7$.

35. Given the function $f(x) = 8 - 3x$:
- **a.** Evaluate $f(-2)$.
- **b.** Solve $f(x) = -1$.

36. Given the function $p(c) = c^2 + c$:
- **a.** Evaluate $p(-3)$.
- **b.** Solve $p(c) = 2$.

37. Given the function $f(x) = x^2 - 3x$
- **a.** Evaluate $f(5)$.
- **b.** Solve $f(x) = 4$

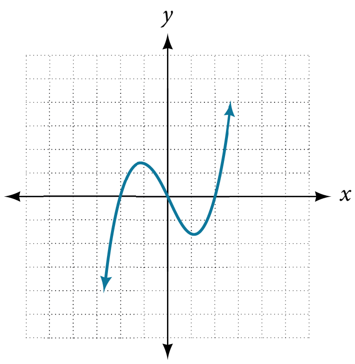
38. Given the function $f(x) = \sqrt{x + 2}$:
- **a.** Evaluate $f(7)$.
- **b.** Solve $f(x) = 4$

39. Consider the relationship $3r + 2t = 18$.
- **a.** Write the relationship as a function $r = f(t)$.
- **b.** Evaluate $f(-3)$.
- **c.** Solve $f(t) = 2$.

GRAPHICAL

For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

40.



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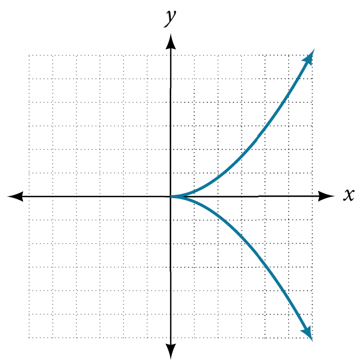

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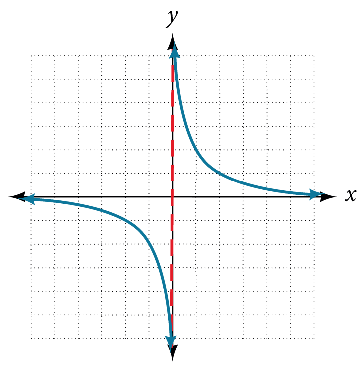

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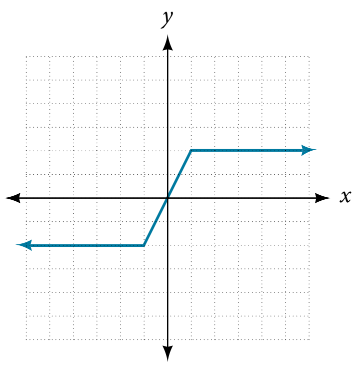

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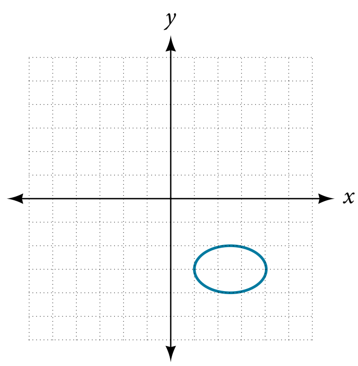

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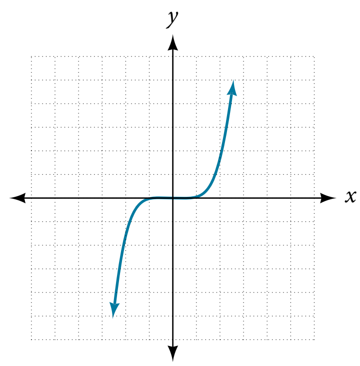

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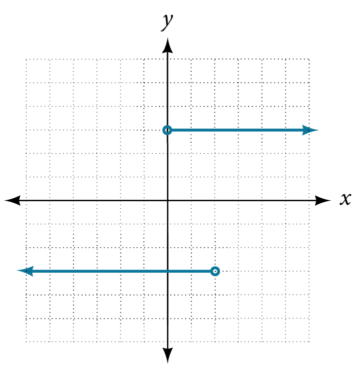
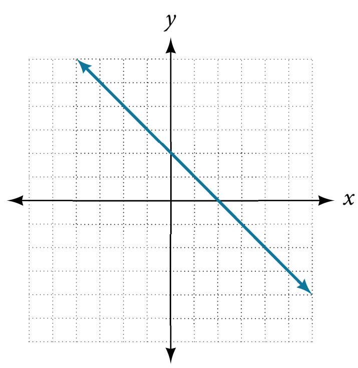
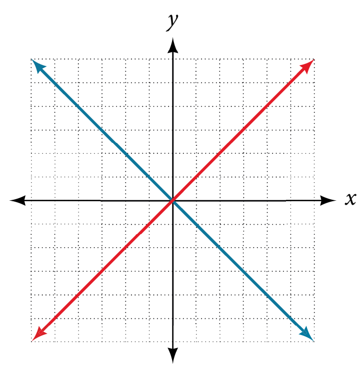


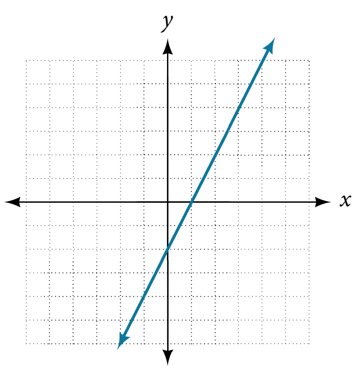
47.



48.



49.



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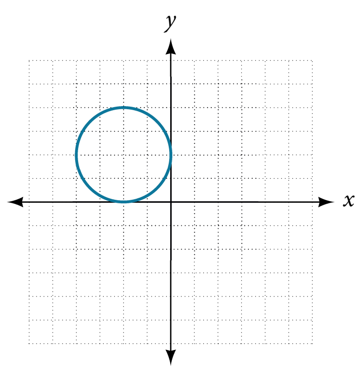
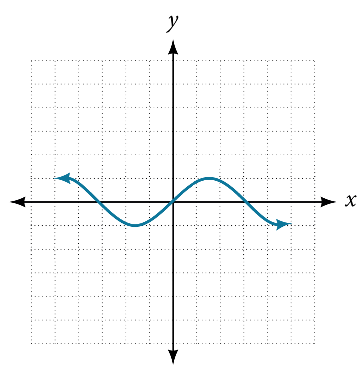


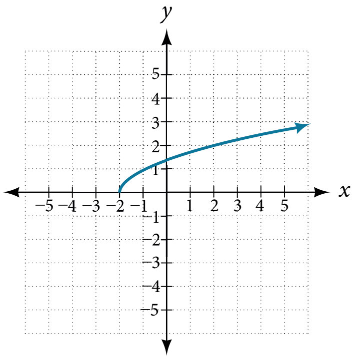
51.



52. Given the following graph

a. Evaluate $f(-1)$.

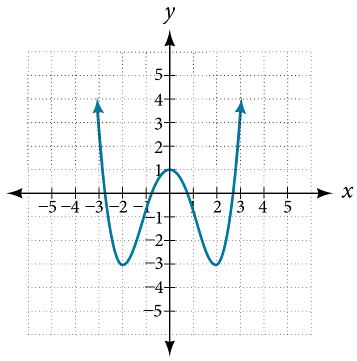
b. Solve for $f(x) = 3$.



53. Given the following graph

a. Evaluate $f(0)$.

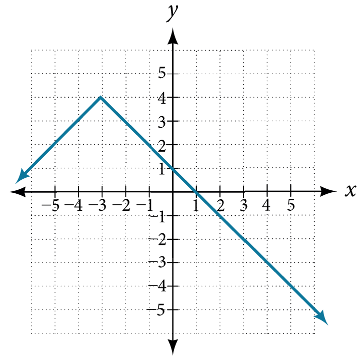
b. Solve for $f(x) = -3$.



54. Given the following graph

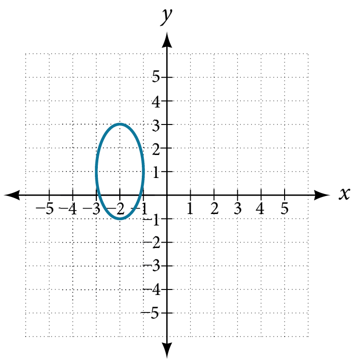
a. Evaluate $f(4)$.

b. Solve for $f(x) = 1$.

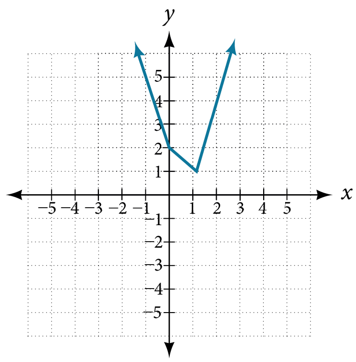


For the following exercises, determine if the given graph is a one-to-one function.

55.



56.



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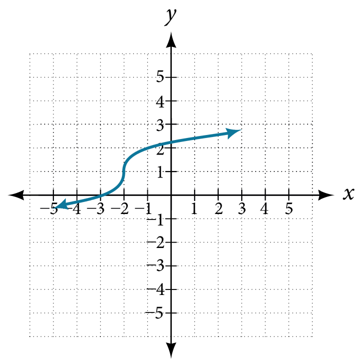
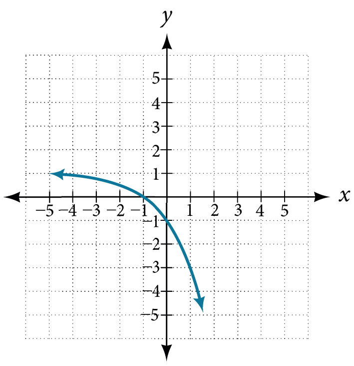
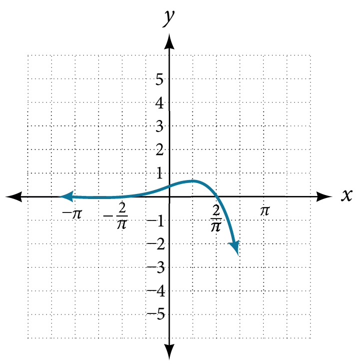


58.



59.



NUMERIC

For the following exercises, determine whether the relation represents a function.

60. $\{(-1, -1),(-2, -2),(-3, -3)\}$

61. $\{(3, 4),(4, 5),(5, 6)\}$

62. $\{(2, 5),(7, 11),(15, 8),(7, 9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

63.

x	5	10	15
y	3	8	14

64.

x	5	10	15
y	3	8	8

65.

x	5	10	10
y	3	8	14

For the following exercises, use the function f represented in **Table 14** below.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

Table 14

66. Evaluate $f(3)$.

67. Solve $f(x) = 1$

For the following exercises, evaluate the function f at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$.

68. $f(x) = 4 - 2x$

69. $f(x) = 8 - 3x$

70. $f(x) = 8x^2 - 7x + 3$

71. $f(x) = 3 + \sqrt{x + 3}$

72. $f(x) = \dfrac{x - 2}{x + 3}$

73. $f(x) = 3^x$

For the following exercises, evaluate the expressions, given functions f, g, and h:

$$f(x) = 3x - 2 \qquad g(x) = 5 - x^2 \qquad h(x) = -2x^2 + 3x - 1$$

74. $3f(1) - 4g(-2)$

75. $f\left(\frac{7}{3}\right) - h(-2)$

TECHNOLOGY

For the following exercises, graph $y = x^2$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

76. $[-0.1, 0.1]$

77. $[-10, 10]$

78. $[-100, 100]$

For the following exercises, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

79. $[-0.1, 0.1]$

80. $[-10, 10]$

81. $[-100, 100]$

For the following exercises, graph $y = \sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

82. $[0, 0.01]$

83. $[0, 100]$

84. $[0, 10{,}000]$

For the following exercises, graph $y = \sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

85. $[-0.001, 0.001]$

86. $[-1{,}000, 1{,}000]$

87. $[-1{,}000{,}000, 1{,}000{,}000]$

REAL-WORLD APPLICATIONS

88. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.

b. Explain the meaning of the statement $f(5) = 2$.

89. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

a. A garden with area 5,000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.

b. Explain the meaning of the statement $g(100) = 1$.

90. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

a. $f(5) = 30$

b. $f(10) = 40$

91. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

a. $h(1) = 200$

b. $h(2) = 350$

92. Show that the function $f(x) = 3(x - 5)^2 + 7$ is <u>not</u> one-to-one.

LEARNING OBJECTIVES

In this section, you will:

- Find the domain of a function defined by an equation.
- Graph piecewise-defined functions.

3.2 DOMAIN AND RANGE

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—*I am Legend*, *Hannibal*, *The Ring*, *The Grudge*, and *The Conjuring*. **Figure 1** shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions such as these.

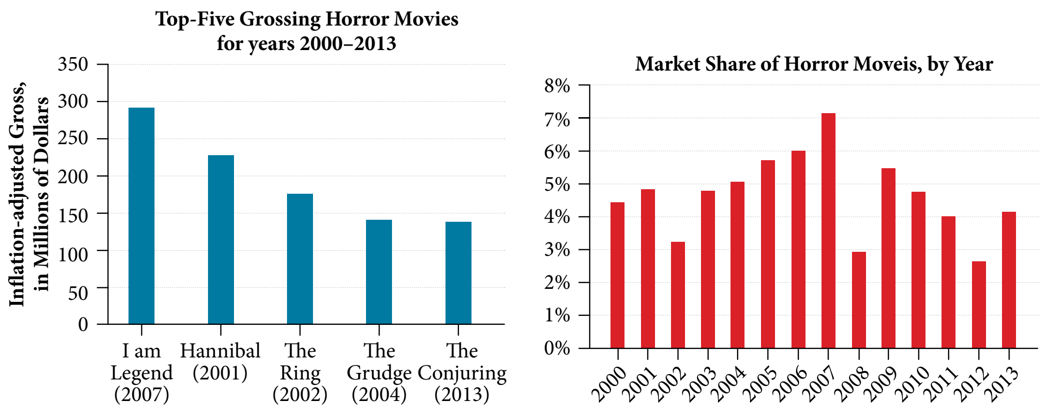


Figure 1 Based on data compiled by www.the-numbers.com.[3]

Finding the Domain of a Function Defined by an Equation

In **Functions and Function Notation**, we were introduced to the concepts of domain and range. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products. See **Figure 2**.

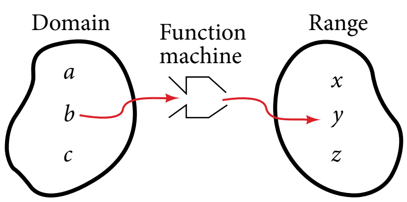


Figure 2

We can write the domain and range in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write (0, 100]. We will discuss interval notation in greater detail later.

3 The Numbers: Where Data and the Movie Business Meet. "Box Office History for Horror Movies." http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:

- The smallest number from the interval is written first.
- The largest number in the interval is written second, following a comma.
- Parentheses, (or), are used to signify that an endpoint value is not included, called exclusive.
- Brackets, [or], are used to indicate that an endpoint value is included, called inclusive.

See **Figure 3** for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty$, a)		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

Figure 3

Example 1 Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain of the following function: {(2, 10), (3, 10), (4, 20), (5, 30), (6, 40)}.

Solution First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$\{2, 3, 4, 5, 6\}$$

Try It #1

Find the domain of the function: $\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$

How To...

Given a function written in equation form, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

Example 2 Finding the Domain of a Function

Find the domain of the function $f(x) = x^2 - 1$.

Solution The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty, \infty)$.

Try It #2

Find the domain of the function: $f(x) = 5 - x + x^3$.

How To…

Given a function written in an equation form that includes a fraction, find the domain.

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x. If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

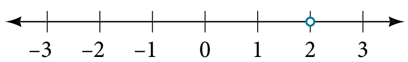
Example 3 Finding the Domain of a Function Involving a Denominator

Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$ as shown in **Figure 4**. We can use a symbol known as the union, $\cup$, to combine the two sets. In interval notation, we write the solution: $(-\infty, 2) \cup (2, \infty)$.



$$x < 2 \text{ or } x > 2$$
$$\downarrow \qquad \downarrow$$
$$(-\infty, 2) \cup (2, \infty)$$

Figure 4

Try It #3

Find the domain of the function: $f(x) = \dfrac{1+4x}{2x-1}$.

How To…

Given a function written in equation form including an even root, find the domain.

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 4 Finding the Domain of a Function with an Even Root

Find the domain of the function $f(x) = \sqrt{7 - x}$.

Solution When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$7 - x \geq 0$$
$$-x \geq -7$$
$$x \leq 7$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

Try It #4

Find the domain of the function $f(x) = \sqrt{5 + 2x}$.

Q & A…

Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\frac{1}{\sqrt{x}}$ has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Notations to Specify Domain and Range

In the previous examples, we used inequalities and lists to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the behavior of the variable in **set-builder notation**. For example, $\{x \mid 10 \leq x < 30\}$ describes the behavior of x in set-builder notation. The braces { } are read as "the set of," and the vertical bar | is read as "such that," so we would read $\{x \mid 10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure 5 compares inequality notation, set-builder notation, and interval notation.

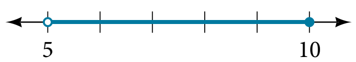
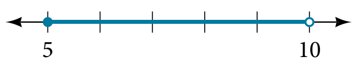
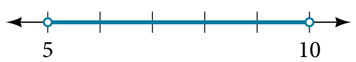
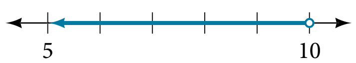
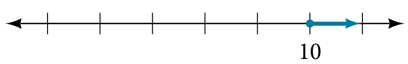
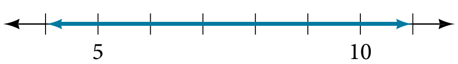
	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	$(5, 10]$
	$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

Figure 5

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, $\cup$, to combine two unconnected intervals. For example, the union of the sets $\{2, 3, 5\}$ and $\{4, 6\}$ is the set $\{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one *or* the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x \mid |x| \geq 3\} = (-\infty, -3] \cup [3, \infty)$$

set-builder notation and ***interval notation***

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x \mid \text{statement about } x\}$ which is read as, "the set of all x such that the statement about x is true." For example,

$$\{x \mid 4 < x \leq 12\}$$

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,

$$(4, 12]$$

How To...

Given a line graph, describe the set of values using interval notation.

1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [with each end value to be included in the set (solid dot) or (for each excluded end value (open dot).
3. At the right end of each interval, use] with each end value to be included in the set (filled dot) or) for each excluded end value (open dot).
4. Use the union symbol $\cup$ to combine all intervals into one set.

Example 5 Describing Sets on the Real-Number Line

Describe the intervals of values shown in **Figure 6** using inequality notation, set-builder notation, and interval notation.

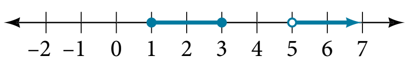


Figure 6

Solution To describe the values, x, included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Inequality	$1 \leq x \leq 3$ or $x > 5$
Set-builder notation	$\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$
Interval notation	$[1, 3] \cup (5, \infty)$

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

Try It #5

Given this figure, specify the graphed set in

a. words

b. set-builder notation

c. interval notation

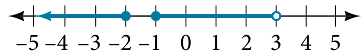


Figure 7

Finding Domain and Range from Graphs

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See **Figure 8**.

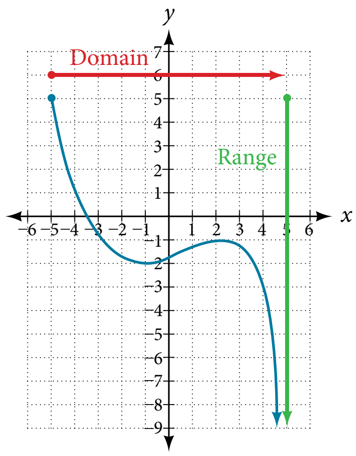


Figure 8

We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 6 Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in **Figure 9**.

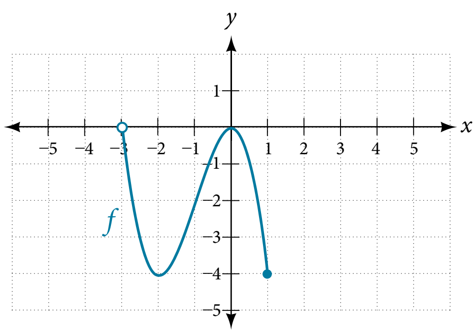


Figure 9

Solution We can observe that the horizontal extent of the graph is −3 to 1, so the domain of f is (−3, 1]. The vertical extent of the graph is 0 to −4, so the range is [−4, 0]. See **Figure 10**.

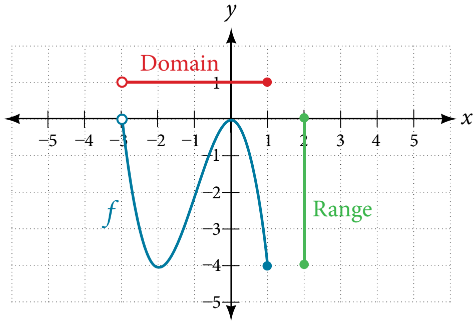


Figure 10

Example 7 Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in **Figure 11.**

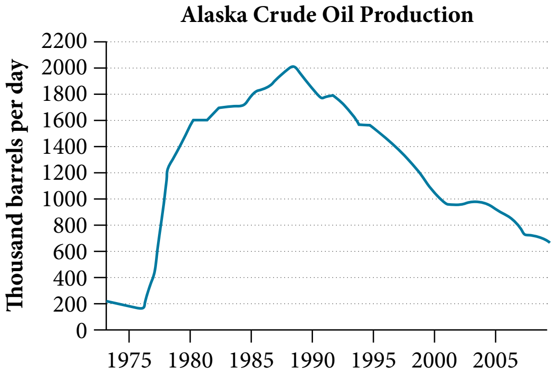


Figure 11 (credit: modification of work by the U.S. Energy Information Administration) [4]

Solution The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \leq t \leq 2008$ and the range as approximately $180 \leq b \leq 2010$.

In interval notation, the domain is [1973, 2008], and the range is about [180, 2010]. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Try It #6

Given **Figure 12**, identify the domain and range using interval notation.

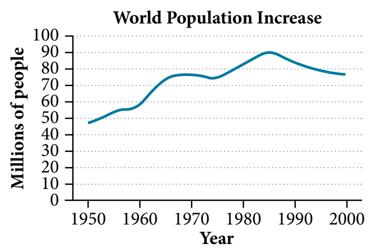


Figure 12

Q & A...

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

4 http://www.eia.gov/dnav/pet/hist/LeafHandler.ashx?n=PET&s=MCRFPAK2&f=A.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

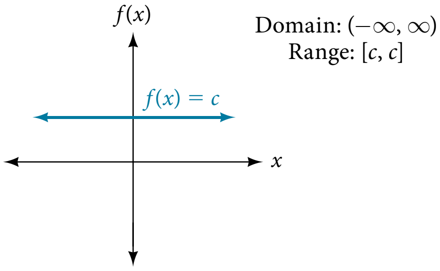


Figure 13

For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this is written as $[c, c]$, the interval that both begins and ends with c.

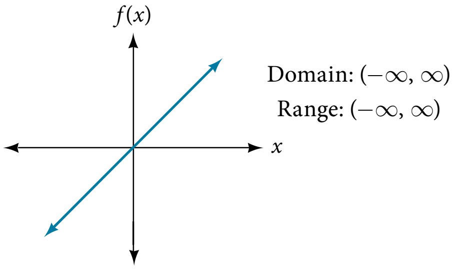


Figure 14

For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.

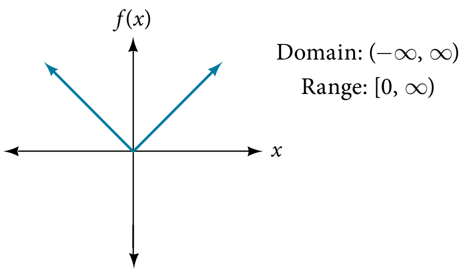


Figure 15

For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

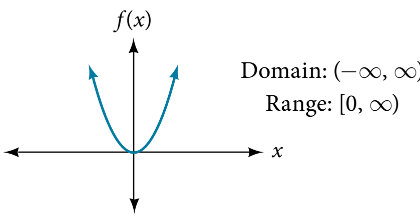


Figure 16

For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.

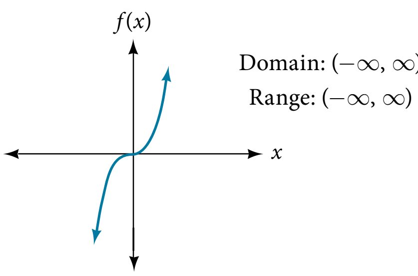


Figure 17

For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.

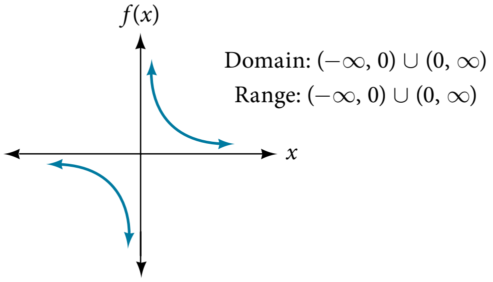


Figure 18

For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x \mid x \neq 0\}$, the set of all real numbers that are not zero.

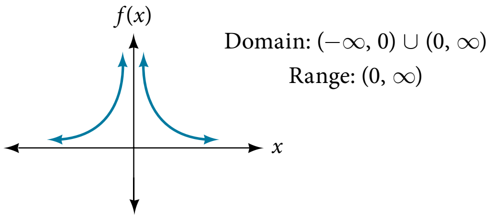


Figure 19

For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$, we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.

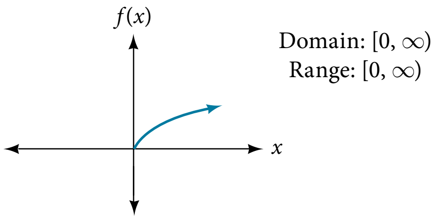


Figure 20

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.

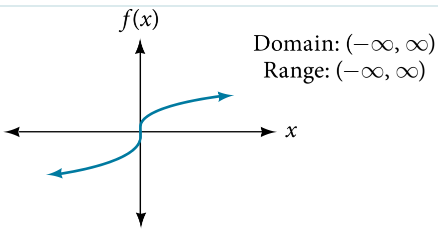


Figure 21

For the **cube root function** $f(x) = \sqrt[3]{x}$ the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

How To...

Given the formula for a function, determine the domain and range.

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

Example 8 Finding the Domain and Range Using Toolkit Functions

Find the domain and range of $f(x) = 2x^3 - x$.

Solution There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result. The domain is $(-\infty, \infty)$ and the range is also $(-\infty, \infty)$.

Example 9 Finding the Domain and Range

Find the domain and range of $f(x) = \dfrac{2}{x+1}$.

Solution We cannot evaluate the function at -1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$.

Example 10 Finding the Domain and Range

Find the domain and range of $f(x) = 2\sqrt{x+4}$.

Solution We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$$x + 4 \geq 0 \text{ when } x \geq -4$$

The domain of $f(x)$ is $[-4, \infty)$.

We then find the range. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis ***Figure 22*** *represents the function f.*

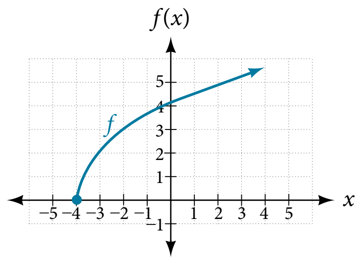


Figure 22

Try It #7

Find the domain and range of $f(x) = -\sqrt{2-x}$.

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, absolute value can be defined as the magnitude, or modulus, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A **piecewise function** is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10{,}000$ and $\$1000 + 0.2(S - \$10{,}000)$ if $S > \$10{,}000$.

piecewise function

A **piecewise function** is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if } x \text{ is in domain 1} \\ \text{formula 2} & \text{if } x \text{ is in domain 2} \\ \text{formula 3} & \text{if } x \text{ is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

How To...

Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example 11 Writing a Piecewise Function

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a function relating the number of people, n, to the cost, C.

Solution Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } 0 < n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Analysis *The function is represented in* ***Figure 23****. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.*

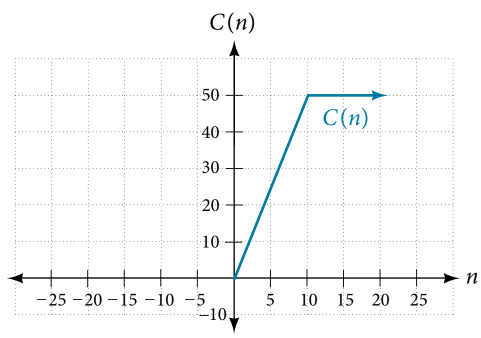


Figure 23

Example 12 Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g - 2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Solution To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4 - 2) = \$45$$

Analysis *The function is represented in* ***Figure 24****. We can see where the function changes from a constant to a shifted and stretched identity at* $g = 2$*. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.*

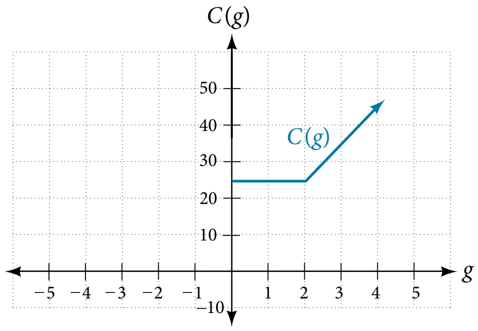


Figure 24

How To…

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 13 Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } \quad x \leq 1 \\ 3 & \text{if } \; 1 < x \leq 2 \\ x & \text{if } \quad x > 2 \end{cases}$$

Solution Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 25 shows the three components of the piecewise function graphed on separate coordinate systems.

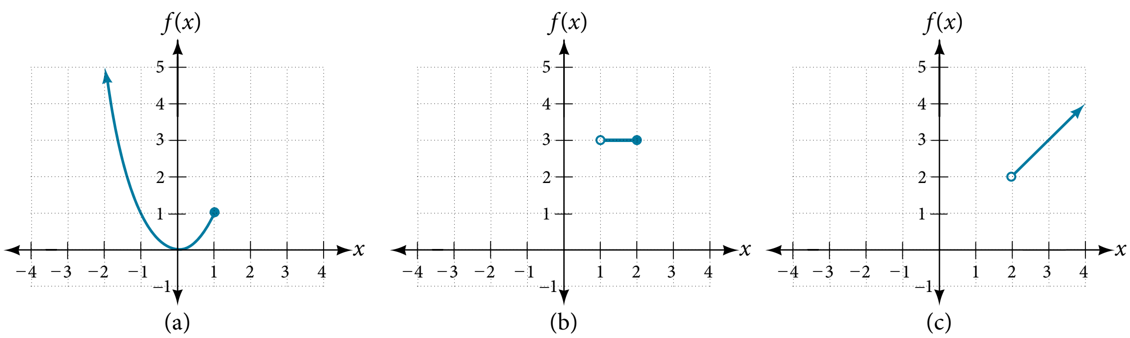


Figure 25 (a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See **Figure 26**.

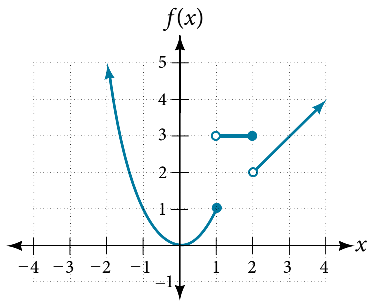


Figure 26

Analysis *Note that the graph does pass the vertical line test even at* $x = 1$ *and* $x = 2$ *because the points* $(1, 3)$ *and* $(2, 2)$ *are not part of the graph of the function, though* $(1, 1)$ *and* $(2, 3)$ *are.*

Try It #8

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if} \quad x < -1 \\ -2 & \text{if} \ -1 < x < 4 \\ \sqrt{x} & \text{if} \quad x > 4 \end{cases}$$

Q & A...

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Access these online resources for additional instruction and practice with domain and range.

- Domain and Range of Square Root Functions (http://openstaxcollege.org/l/domainsqroot)
- Determining Domain and Range (http://openstaxcollege.org/l/determinedomain)
- Find Domain and Range Given the Graph (http://openstaxcollege.org/l/drgraph)
- Find Domain and Range Given a Table (http://openstaxcollege.org/l/drtable)
- Find Domain and Range Given Points on a Coordinate Plane (http://openstaxcollege.org/l/drcoordinate)

3.2 SECTION EXERCISES

VERBAL

1. Why does the domain differ for different functions?

2. How do we determine the domain of a function defined by an equation?

3. Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

4. When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

5. How do you graph a piecewise function?

ALGEBRAIC

For the following exercises, find the domain of each function using interval notation.

6. $f(x) = -2x(x - 1)(x - 2)$

7. $f(x) = 5 - 2x^2$

8. $f(x) = 3\sqrt{x - 2}$

9. $f(x) = 3 - \sqrt{6 - 2x}$

10. $f(x) = \sqrt{4 - 3x}$

11. $f(x) = \sqrt{x^2 + 4}$

12. $f(x) = \sqrt[3]{1 - 2x}$

13. $f(x) = \sqrt[3]{x - 1}$

14. $f(x) = \dfrac{9}{x - 6}$

15. $f(x) = \dfrac{3x + 1}{4x + 2}$

16. $f(x) = \dfrac{\sqrt{x + 4}}{x - 4}$

17. $f(x) = \dfrac{x - 3}{x^2 + 9x - 22}$

18. $f(x) = \dfrac{1}{x^2 - x - 6}$

19. $f(x) = \dfrac{2x^3 - 250}{x^2 - 2x - 15}$

20. $f(x) = \dfrac{5}{\sqrt{x - 3}}$

21. $f(x) = \dfrac{2x + 1}{\sqrt{5 - x}}$

22. $f(x) = \dfrac{\sqrt{x - 4}}{\sqrt{x - 6}}$

23. $f(x) = \dfrac{\sqrt{x - 6}}{\sqrt{x - 4}}$

24. $f(x) = \dfrac{x}{x}$

25. $f(x) = \dfrac{x^2 - 9x}{x^2 - 81}$

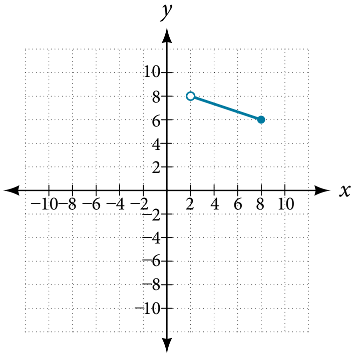
26. Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

- **a.** using algebra.
- **b.** graphing the function in the radicand and determining intervals on the x-axis for which the radicand is nonnegative.

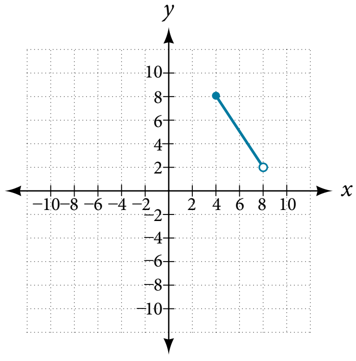
GRAPHICAL

For the following exercises, write the domain and range of each function using interval notation.

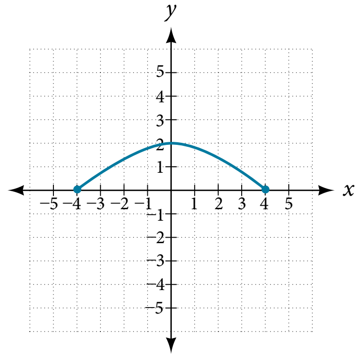
27.



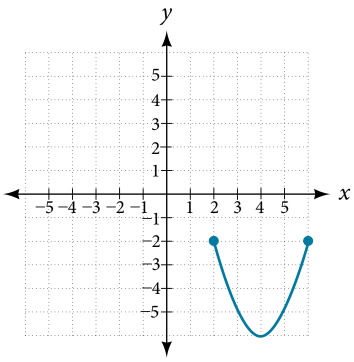
28.



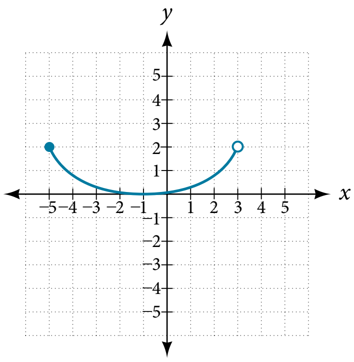
29.



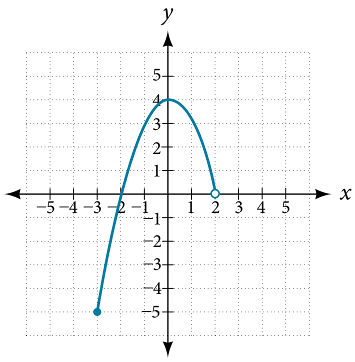
30.



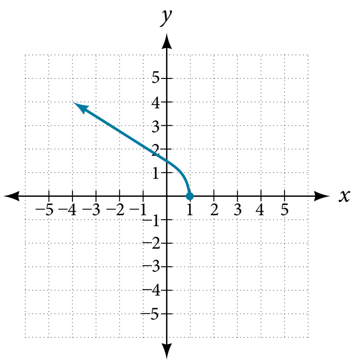
31.



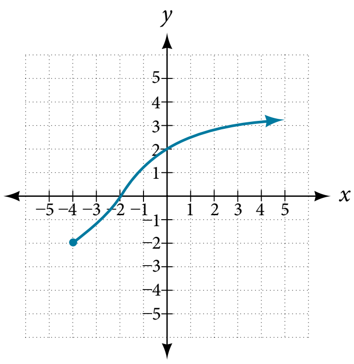
32.



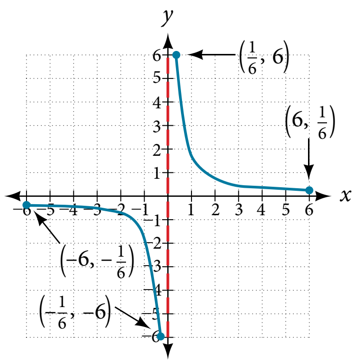
33.



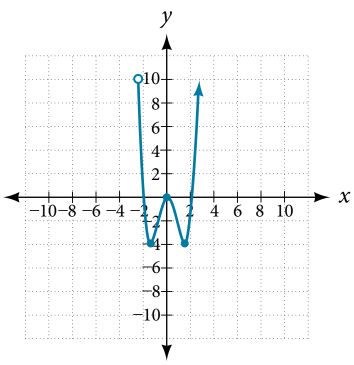
34.



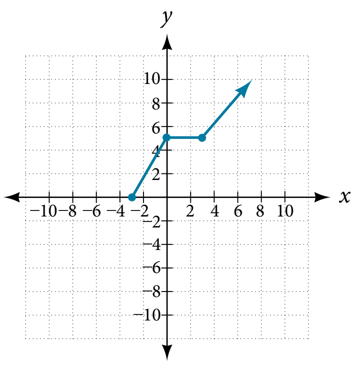
35.



36.



37.



For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

38. $f(x)=\begin{cases} x+1 & \text{if } x<-2 \\ -2x-3 & \text{if } x\geq -2 \end{cases}$

39. $f(x)=\begin{cases} 2x-1 & \text{if } x<1 \\ 1+x & \text{if } x\geq 1 \end{cases}$

40. $f(x)=\begin{cases} x+1 & \text{if } x<0 \\ x-1 & \text{if } x>0 \end{cases}$

41. $f(x)=\begin{cases} 3 & \text{if } x<0 \\ \sqrt{x} & \text{if } x\geq 0 \end{cases}$

42. $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ 1-x & \text{if } x>0 \end{cases}$

43. $f(x)=\begin{cases} x^2 & \text{if } x<0 \\ x+2 & \text{if } x\geq 0 \end{cases}$

44. $f(x)=\begin{cases} x+1 & \text{if } x<1 \\ x^3 & \text{if } x\geq 1 \end{cases}$

45. $f(x)=\begin{cases} |x| & \text{if } x<2 \\ 1 & \text{if } x\geq 2 \end{cases}$

NUMERIC

For the following exercises, given each function f, evaluate $f(-3), f(-2), f(-1)$, and $f(0)$.

46. $f(x) = \begin{cases} x+1 & \text{if} \quad x < -2 \\ -2x-3 & \text{if} \quad x \geq -2 \end{cases}$

47. $f(x) = \begin{cases} 1 & \text{if} \ x \leq -3 \\ 0 & \text{if} \ x > -3 \end{cases}$

48. $f(x) = \begin{cases} -2x^2+3 & \text{if} \ x \leq -1 \\ 5x-7 & \text{if} \ x > -1 \end{cases}$

For the following exercises, given each function f, evaluate $f(-1), f(0), f(2)$, and $f(4)$.

49. $f(x) = \begin{cases} 7x+3 & \text{if} \ x < 0 \\ 7x+6 & \text{if} \ x \geq 0 \end{cases}$

50. $f(x) = \begin{cases} x^2-2 & \text{if} \ x < 2 \\ 4+|x-5| & \text{if} \ x \geq 2 \end{cases}$

51. $f(x) = \begin{cases} 5x & \text{if} \quad x < 0 \\ 3 & \text{if} \quad 0 \leq x \leq 3 \\ x^2 & \text{if} \quad x > 3 \end{cases}$

For the following exercises, write the domain for the piecewise function in interval notation.

52. $f(x) = \begin{cases} x+1 & \text{if} \quad x < -2 \\ -2x-3 & \text{if} \quad x \geq -2 \end{cases}$

53. $f(x) = \begin{cases} x^2-2 & \text{if} \ x < 1 \\ -x^2+2 & \text{if} \ x > 1 \end{cases}$

54. $f(x) = \begin{cases} 2x-3 & \text{if} \ x < 0 \\ -3x^2 & \text{if} \ x \geq 2 \end{cases}$

TECHNOLOGY

55. Graph $y = \dfrac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

56. Graph $y = \dfrac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

EXTENSION

57. Suppose the range of a function f is $[-5, 8]$. What is the range of $|f(x)|$?

58. Create a function in which the range is all nonnegative real numbers.

59. Create a function in which the domain is $x > 2$.

REAL-WORLD APPLICATIONS

60. The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

61. The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.

b. What is the cost of making 25 items?

c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

LEARNING OBJECTIVES

In this section, you will:

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

3.3 RATES OF CHANGE AND BEHAVIOR OF GRAPHS

Gasoline costs have experienced some wild fluctuations over the last several decades. **Table 1**[5] lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

y	2005	2006	2007	2008	2009	2010	2011	2012
$C(y)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

Table 1

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \$2.31 to \$3.68, an increase of \$1.37. While this is interesting, it might be more useful to look at how much the price changed *per year*. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in **Table 1** did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}}\\ &= \frac{\Delta y}{\Delta x}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by \$1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7 \text{ years}} \approx 0.196 \text{ dollars per year}$$

On average, the price of gas increased by about 19.6¢ each year.

Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by \$4,000 per quarter

5 http://www.eia.gov/totalenergy/data/annual/showtext.cfm?t=ptb0524. Accessed 3/5/2014.

rate of change

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

How To…

Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2.

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Example 1 Computing an Average Rate of Change

Using the data in **Table 1**, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{\$2.41 - \$2.84}{2009 - 2007}\\ &= \frac{-\$0.43}{2 \text{ years}}\\ &= -\$0.22 \text{ per year}\end{aligned}$$

Analysis *Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.*

Try It #1

Using the data in **Table 1** at the beginning of this section, find the average rate of change between 2005 and 2010.

Example 2 Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in **Figure 1**, find the average rate of change on the interval $[-1, 2]$.

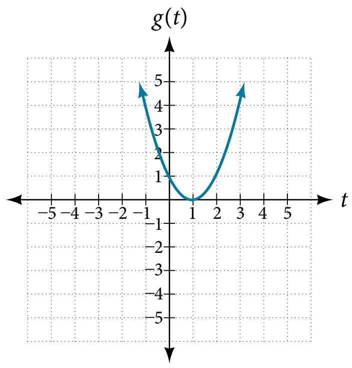


Figure 1

Solution At $t = -1$, **Figure 2** shows $g(-1) = 4$. At $t = 2$, the graph shows $g(2) = 1$.

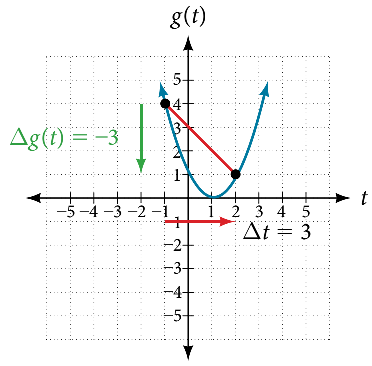


Figure 2

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The average rate of change is shown by the slope of the red line segment. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1$$

Analysis *Note that the order we choose is very important. If, for example, we use* $\frac{y_2 - y_1}{x_1 - x_2}$, *we will not get the correct answer. Decide which point will be* 1 *and which point will be* 2, *and keep the coordinates fixed as* (x_1, y_1) *and* (x_2, y_2).

Example 3 Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away and leaving on a trip, Anna records her distance from home over time. The values are shown in **Table 2**. Find her average speed over the first 6 hours.

***t* (hours)**	0	1	2	3	4	5	6	7
***D*(*t*) (miles)**	10	55	90	153	214	240	282	300

Table 2

Solution Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours.

$$\frac{292-10}{6-0} = \frac{282}{6}$$
$$= 47$$

The average speed is 47 miles per hour.

Analysis *Because the speed is not constant, the average speed depends on the interval chosen. For the interval* [2, 3], *the average speed is 63 miles per hour.*

Example 4 Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4].

Solution We can start by computing the function values at each endpoint of the interval.

$$f(2) = 2^2 - \frac{1}{2} \qquad f(4) = 4^2 - \frac{1}{4}$$
$$= 4 - \frac{1}{2} \qquad = 16 - \frac{1}{4}$$
$$= \frac{7}{2} \qquad = \frac{63}{4}$$

Now we compute the average rate of change.

$$\begin{aligned}\text{Average rate of change} &= \frac{f(4) - f(2)}{4 - 2}\\ &= \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2}\\ &= \frac{\frac{49}{4}}{2}\\ &= \frac{49}{8}\end{aligned}$$

Try It #2

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval [1, 9].

Example 5 Finding the Average Rate of Change of a Force

The electrostatic force F, measured in newtons, between two charged particles can be related to the distance between the particles d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval [2, 6].

$$\begin{aligned}\text{Average rate of change} &= \frac{F(6) - F(2)}{6 - 2}\\ &= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{6 - 2} && \text{Simplify.}\\ &= \frac{\frac{2}{36} - \frac{2}{4}}{4}\\ &= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms.}\\ &= -\frac{1}{9} && \text{Simplify.}\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 6 Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a in simplest form.

Solution We use the average rate of change formula.

$$\begin{aligned}\text{Average rate of change} &= \frac{g(a) - g(0)}{a - 0} && \text{Evaluate.}\\ &= \frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0} && \text{Simplify.}\\ &= \frac{a^2 + 3a + 1 - 1}{a} && \text{Simplify and factor.}\\ &= \frac{a(a + 3)}{a} && \text{Divide by the common factor } a.\\ &= a + 3\end{aligned}$$

This result tells us the average rate of change in terms of a between $t = 0$ and any other point $t = a$. For example, on the interval [0, 5], the average rate of change would be $5 + 3 = 8$.

Try It #3

Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$ in simplest forms in terms of a.

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. **Figure 3** shows examples of increasing and decreasing intervals on a function.

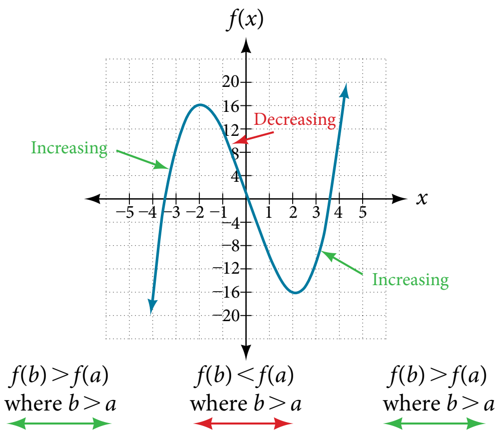


Figure 3 The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term *local* is replaced by the term *relative*. In this text, we will use the term *local*.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of *local* extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in **Figure 4**, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

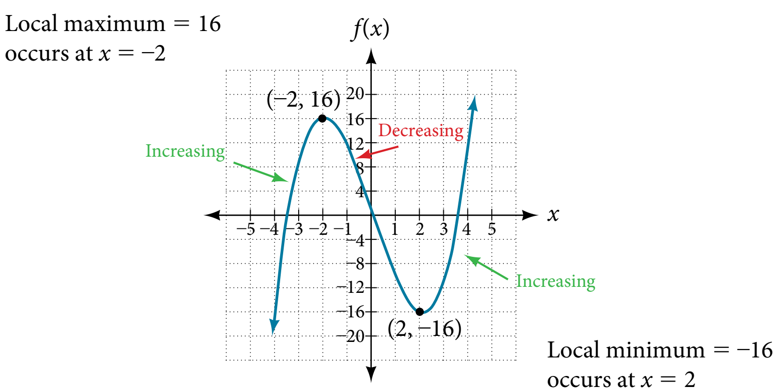


Figure 4

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. **Figure 5** illustrates these ideas for a local maximum.

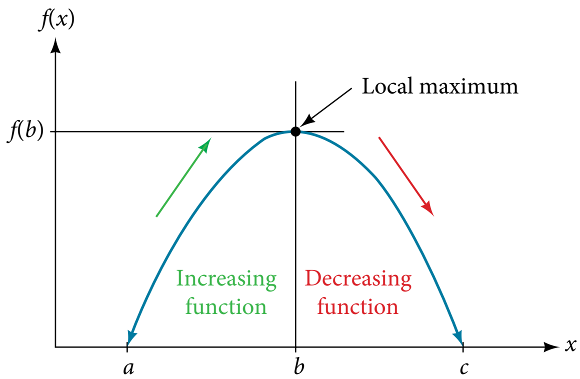


Figure 5 Definition of a local maximum

These observations lead us to a formal definition of local extrema.

> ***local minima and local maxima***
>
> A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$.
>
> A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$.
>
> A function f has a local maximum at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \leq f(b)$. Likewise, f has a local minimum at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \geq f(b)$.

Example 7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in **Figure 6**, identify the intervals on which the function appears to be increasing.

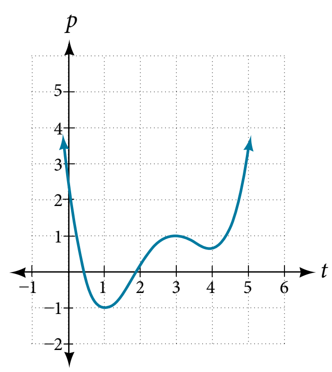


Figure 6

Solution We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis *Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).*

Example 8 Finding Local Extrema from a Graph

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution Using technology, we find that the graph of the function looks like that in **Figure 7**. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$.

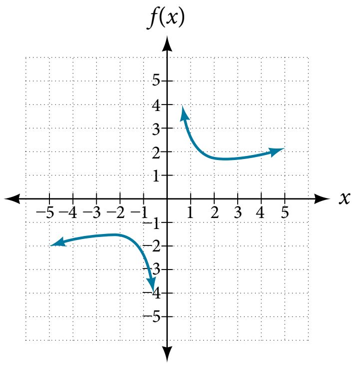


Figure 7

Analysis *Most graphing calculators and graphing utilities can estimate the location of maxima and minima.* ***Figure 8*** *provides screen images from two different technologies, showing the estimate for the local maximum and minimum.*

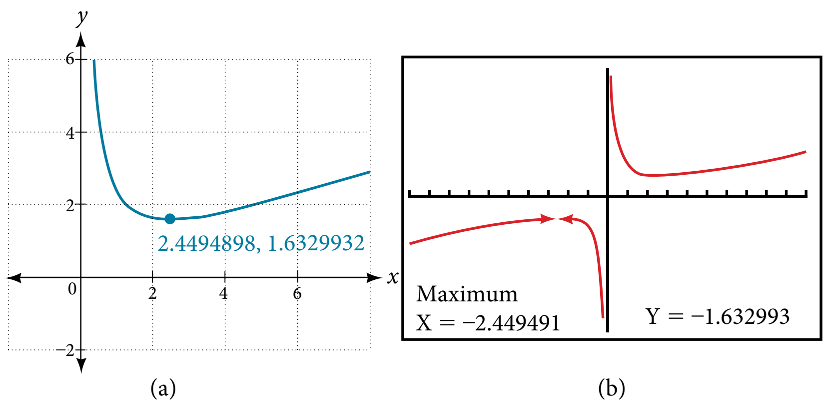


Figure 8

Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

Try It #4

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Example 9 Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in **Figure 9**, find all local maxima and minima.

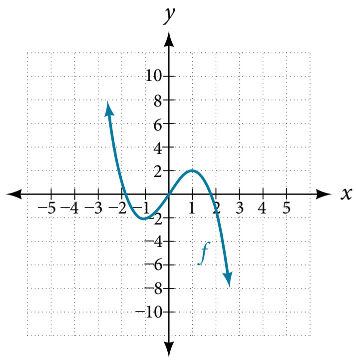


Figure 9

Solution Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$. The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$.

The local minimum is the y-coordinate at $x = -1$, which is -2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in **Figure 10**, **Figure 11**, and **Figure 12**.

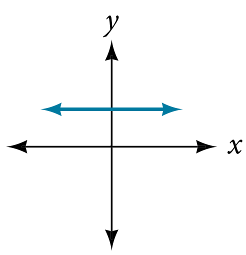
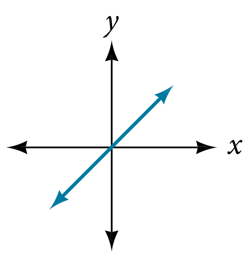
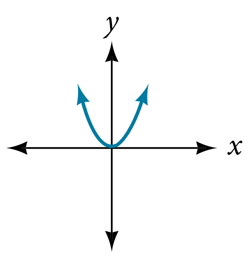
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 10

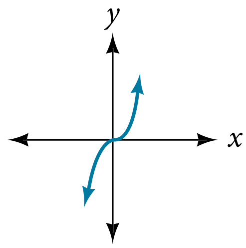
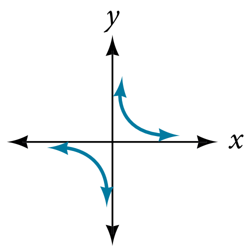
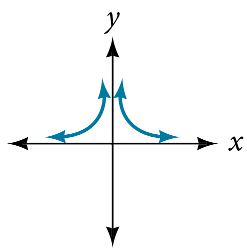
Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

Figure 11

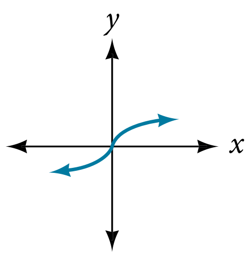
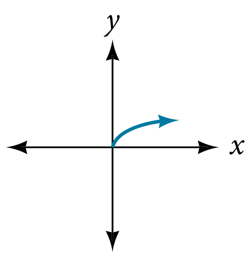
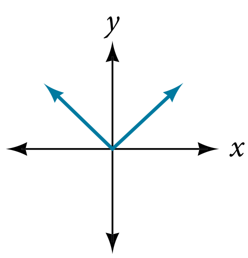
Function	Increasing/Decreasing	Example
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	

Figure 12

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function. See **Figure 13**.

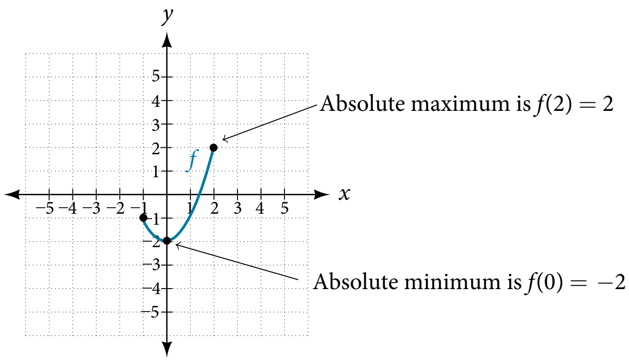


Figure 13

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

absolute maxima and minima

The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.

The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

Example 10 **Finding Absolute Maxima and Minima from a Graph**

For the function f shown in **Figure 14**, find all absolute maxima and minima.

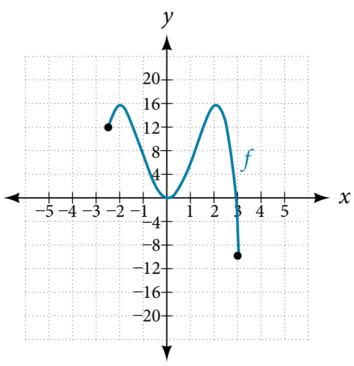


Figure 14

Solution Observe the graph of f. The graph attains an absolute maximum in two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an absolute minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the y-coordinate at $x = 3$, which is -10.

Access this online resource for additional instruction and practice with rates of change.

- Average Rate of Change (http://openstaxcollege.org/l/aroc)

3.3 SECTION EXERCISES

VERBAL

1. Can the average rate of change of a function be constant?

2. If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?

3. How are the absolute maximum and minimum similar to and different from the local extrema?

4. How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

ALGEBRAIC

For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h in simplest form.

5. $f(x) = 4x^2 - 7$ on $[1, b]$

6. $g(x) = 2x^2 - 9$ on $[4, \text{b}]$

7. $p(x) = 3x + 4$ on $[2, 2 + h]$

8. $k(x) = 4x - 2$ on $[3, 3 + h]$

9. $f(x) = 2x^2 + 1$ on $[x, x + h]$

10. $g(x) = 3x^2 - 2$ on $[x, x + h]$

11. $a(t) = \dfrac{1}{t + 4}$ on $[9, 9 + h]$

12. $b(x) = \dfrac{1}{x + 3}$ on $[1, 1 + h]$

13. $j(x) = 3x^3$ on $[1, 1 + h]$

14. $r(t) = 4t^3$ on $[2, 2 + h]$

15. $\dfrac{f(x + h) - f(x)}{h}$ given $f(x) = 2x^2 - 3x$ on $[x, x + h]$

GRAPHICAL

For the following exercises, consider the graph of f shown in **Figure 15**.

16. Estimate the average rate of change from $x = 1$ to $x = 4$.

17. Estimate the average rate of change from $x = 2$ to $x = 5$.

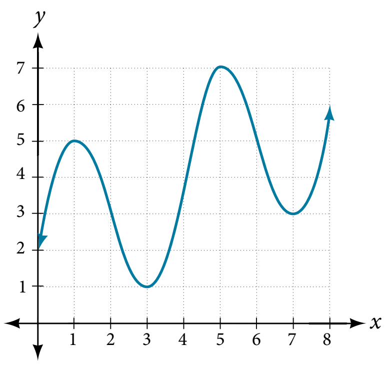
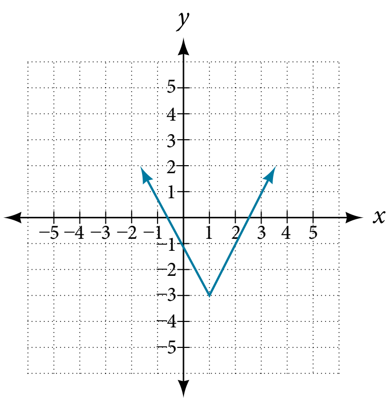


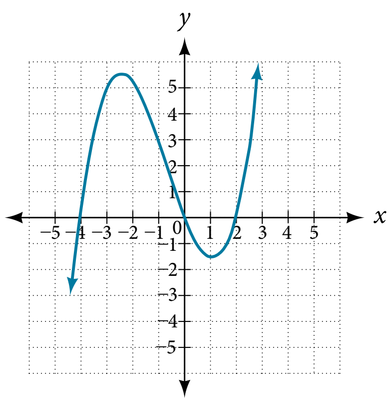
Figure 15

For the following exercises, use the graph of each function to estimate the intervals on which the function is increasing or decreasing.

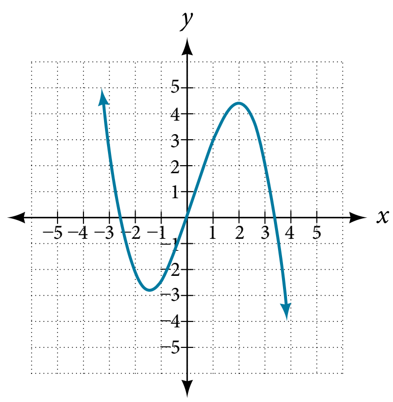
18.



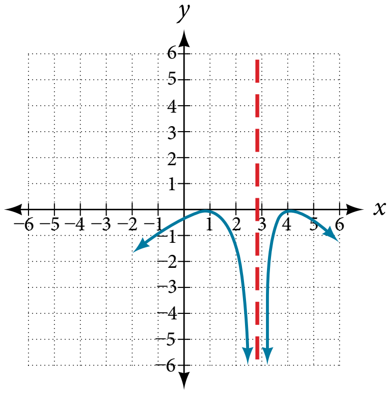
19.



20.



21.



For the following exercises, consider the graph shown in **Figure 16**.

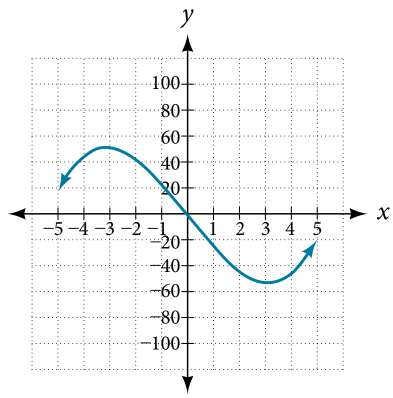


Figure 16

22. Estimate the intervals where the function is increasing or decreasing.

23. Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

For the following exercises, consider the graph in **Figure 17.**

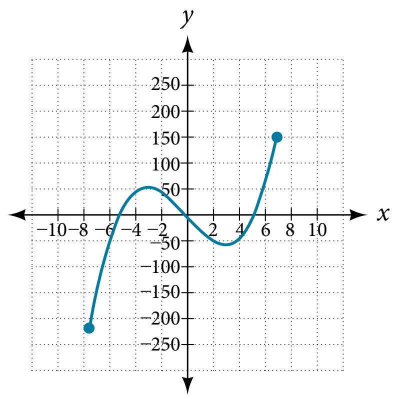


Figure 17

24. If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

25. If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

NUMERIC

26. Table 3 gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (**a**) between 2001 and 2002, and (**b**) between 2001 and 2004?

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006
Sales (millions of dollars)	201	219	233	243	249	251	249	243	233

Table 3

27. Table 4 gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (**a**) between 2002 and 2004, and (**b**) between 2002 and 2006?

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008
Population (thousands)	87	84	83	80	77	76	78	81	85

Table 4

For the following exercises, find the average rate of change of each function on the interval specified.

28. $f(x) = x^2$ on $[1, 5]$

29. $h(x) = 5 - 2x^2$ on $[-2, 4]$

30. $q(x) = x^3$ on $[-4, 2]$

31. $g(x) = 3x^3 - 1$ on $[-3, 3]$

32. $y = \frac{1}{x}$ on $[1, 3]$

33. $p(t) = \frac{(t^2 - 4)(t + 1)}{t^2 + 3}$ on $[-3, 1]$

34. $k(t) = 6t^2 + \frac{4}{t^3}$ on $[-1, 3]$

TECHNOLOGY

For the following exercises, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

35. $f(x) = x^4 - 4x^3 + 5$

36. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

37. $g(t) = t\sqrt{t + 3}$

38. $k(t) = 3t^{\frac{2}{3}} - t$

39. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

40. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

EXTENSION

41. The graph of the function f is shown in **Figure 18**.

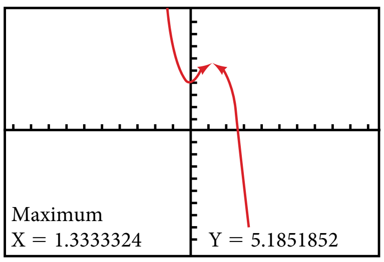


Figure 18

Based on the calculator screen shot, the point (1.333, 5.185) is which of the following?

a. a relative (local) maximum of the function
b. the vertex of the function
c. the absolute maximum of the function
d. a zero of the function

42. Let $f(x) = \frac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$

43. Let $f(x) = \frac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

REAL-WORLD APPLICATIONS

44. At the start of a trip, the odometer on a car read 21,395. At the end of the trip, 13.5 hours later, the odometer read 22,125. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

45. A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?

46. Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

47. The graph in **Figure 19** illustrates the decay of a radioactive substance over t days.

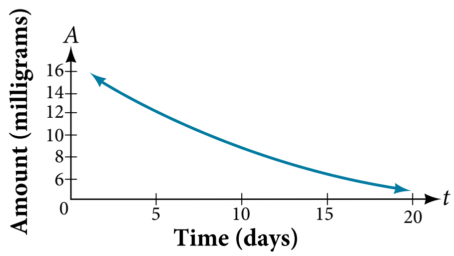


Figure 19

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

LEARNING OBJECTIVES

In this section, you will:

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

3.4 COMPOSITION OF FUNCTIONS

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, Cost $= C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write $C(T(5))$.

$$\overbrace{C(T(5))}^{\text{Cost for the temperature}}$$

$$\uparrow$$

Temperature on day 5

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f + g$, $f - g$, fg, and $\frac{f}{g}$ by the relations

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{where } g(x) \neq 0$$

Example 1 **Performing Algebraic Operations on Functions**

Find and simplify the functions $(g - f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x - 1$ and $g(x) = x^2 - 1$. Are they the same function?

Solution Begin by writing the general form, and then substitute the given functions.

$$(g - f)(x) = g(x) - f(x)$$

$$(g - f)(x) = x^2 - 1 - (x - 1)$$

$$(g - f)(x) = x^2 - x$$

$$(g - f)(x) = x(x - 1)$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)}$$

$$\left(\frac{g}{f}\right)(x) = \frac{x^2 - 1}{x - 1} \qquad \text{where } x \neq 1$$

$$\left(\frac{g}{f}\right)(x) = \frac{(x + 1)(x - 1)}{x - 1} \qquad \text{where } x \neq 1$$

$$\left(\frac{g}{f}\right)(x) = x + 1 \qquad \text{where } x \neq 1$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, the denominator is equal to 0, which makes the function undefined.

Try It #1

Find and simplify the functions $(fg)(x)$ and $(f - g)(x)$.

$$f(x) = x - 1 \text{ and } g(x) = x^2 - 1$$

Are they the same function?

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

$g(x)$, the output of g is the input of f

$$(f \circ g)(x) = f(g(x))$$

x is the input of g

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

> ***composition of functions***
> When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that
>
> $$(f \circ g)(x) = f(g(x))$$
>
> The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f. It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 2 **Determining whether Composition of Functions is Commutative**

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is commutative.

$$f(x) = 2x + 1 \qquad g(x) = 3 - x$$

Solution Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3 - x) + 1 \\ &= 6 - 2x + 1 \\ &= 7 - 2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3 - (2x + 1) \\ &= 3 - 2x - 1 \\ &= -2x + 2 \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative.

Example 3 Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 4 Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles} = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons} = g\,(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Q & A...

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

Try It #2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 5 Using a Table to Evaluate a Composite Function

Using **Table 1**, evaluate $f(g(3))$ and $g(f(3))$.

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Table 1

Solution To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g: $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2$$
$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Table 2

Try It #3

Using **Table 1**, evaluate $f(g(1))$ and $g(f(4))$.

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How To...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 6 Using a Graph to Evaluate a Composite Function

Using **Figure 1**, evaluate $f(g(1))$.

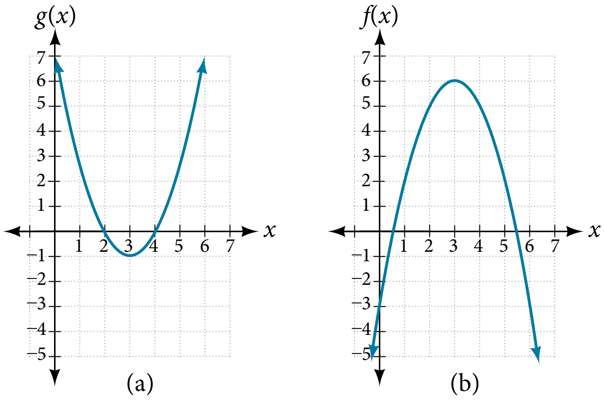


Figure 1

Solution To evaluate $f(g(1))$, we start with the inside evaluation. See **Figure 2**.

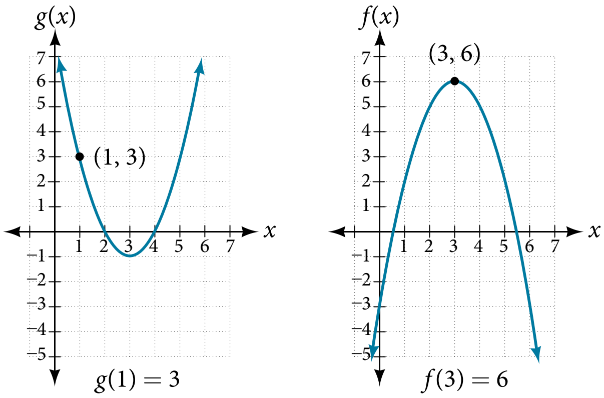


Figure 2

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis ***Figure 3*** *shows how we can mark the graphs with arrows to trace the path from the input value to the output value.*

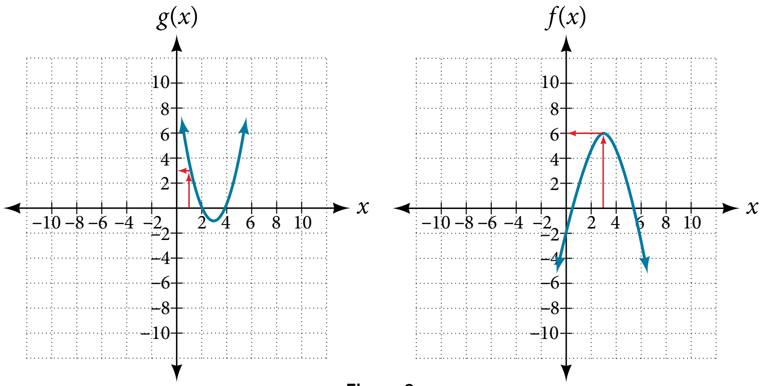


Figure 3

Try It #4

Using **Figure 1**, evaluate $g(f(2))$.

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To…

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 7 Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$

$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$

$$f(h(1)) = 5^2 - 5$$

$$f(h(1)) = 20$$

Analysis *It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.*

Try It #5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$ **b.** $h(f(-2))$

Finding the Domain of a Composite Function

As we discussed previously, the domain of a composite function such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

domain of a composite function

The domain of a composite function $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

How To...

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 8 Finding the Domain of a Composite Function

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\frac{4}{3x-2} = 1$$

$$4 = 3x - 2$$

$$6 = 3x$$

$$x = 2$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example 9 Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \sqrt{x+2} \text{ and } g(x) = \sqrt{3-x}$$

Solution Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x} \geq 0$, $3 - x \geq 0$, which gives a domain of $(-\infty, 3]$.

Analysis *This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.*

Try It #6

Find the domain of $(f \circ g)(x)$ where $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x+4}$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to decompose a composite function, so we may choose the decomposition that appears to be most expedient.

Example 10 Decomposing a Function

Write $f(x) = \sqrt{5-x^2}$ as the composition of two functions.

Solution We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5 - x^2) = \sqrt{5-x^2}$$

Try It #7

Write $f(x) = \dfrac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

3.4 SECTION EXERCISES

VERBAL

1. How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

2. What is the composition of two functions, $f \circ g$?

3. If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

4. How do you find the domain for the composition of two functions, $f \circ g$?

ALGEBRAIC

For the following exercises, determine the domain for each function in interval notation.

5. Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$.

6. Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$.

7. Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$.

8. Given $f(x) = \frac{1}{x - 4}$ and $g(x) = \frac{1}{6 - x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$.

9. Given $f(x) = 3x^2$ and $g(x) = \sqrt{x - 5}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$.

10. Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$.

11. For the following exercise, find the indicated function given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$.

a. $f(g(2))$ **b.** $f(g(x))$ **c.** $g(f(x))$ **d.** $(g \circ g)(x)$ **e.** $(f \circ f)(-2)$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

12. $f(x) = x^2 + 1$, $g(x) = \sqrt{x + 2}$

13. $f(x) = \sqrt{x} + 2$, $g(x) = x^2 + 3$

14. $f(x) = |x|$, $g(x) = 5x + 1$

15. $f(x) = \sqrt[3]{x}$, $g(x) = \frac{x + 1}{x^3}$

16. $f(x) = \frac{1}{x - 6}$, $g(x) = \frac{7}{x} + 6$

17. $f(x) = \frac{1}{x-4}$, $g(x) = \frac{2}{x} + 4$

For the following exercises, use each set of functions to find $f(g(h(x)))$. Simplify your answers.

18. $f(x) = x^4 + 6$, $g(x) = x - 6$, and $h(x) = \sqrt{x}$

19. $f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$, and $h(x) = x + 3$

20. Given $f(x) = \frac{1}{x}$, and $g(x) = x - 3$, find the following:

a. $(f \circ g)(x)$

b. the domain of $(f \circ g)(x)$ in interval notation

c. $(g \circ f)(x)$

d. the domain of $(g \circ f)(x)$

e. $\left(\frac{f}{g}\right)x$

21. Given $f(x) = \sqrt{2 - 4x}$ and $g(x) = -\frac{3}{x}$, find the following:

a. $(g \circ f)(x)$

b. the domain of $(g \circ f)(x)$ in interval notation

22. Given the functions $f(x) = \frac{1-x}{x}$ and $g(x) = \frac{1}{1+x^2}$, find the following:

a. $(g \circ f)(x)$
b. $(g \circ f)(2)$

23. Given functions $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4$, state the domain of each of the following functions using interval notation:

a. $\frac{p(x)}{m(x)}$
b. $p(m(x))$
c. $m(p(x))$

24. Given functions $q(x) = \frac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9$, state the domain of each of the following functions using interval notation.

a. $\frac{q(x)}{h(x)}$
b. $q(h(x))$
c. $h(q(x))$

25. For $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x-1}$, write the domain of $(f \circ g)(x)$ in interval notation.

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(g(x))$.

26. $h(x) = (x+2)^2$

27. $h(x) = (x-5)^3$

28. $h(x) = \frac{3}{x-5}$

29. $h(x) = \frac{4}{(x+2)^2}$

30. $h(x) = 4 + \sqrt[3]{x}$

31. $h(x) = \sqrt[3]{\frac{1}{2x-3}}$

32. $h(x) = \frac{1}{(3x^2-4)^{-3}}$

33. $h(x) = \sqrt[4]{\frac{3x-2}{x+5}}$

34. $h(x) = \left(\frac{8+x^3}{8-x^3}\right)^4$

35. $h(x) = \sqrt{2x+6}$

36. $h(x) = (5x-1)^3$

37. $h(x) = \sqrt[3]{x-1}$

38. $h(x) = |x^2 + 7|$

39. $h(x) = \frac{1}{(x-2)^3}$

40. $h(x) = \left(\frac{1}{2x-3}\right)^2$

41. $h(x) = \sqrt{\frac{2x-1}{3x+4}}$

GRAPHICAL

For the following exercises, use the graphs of f, shown in **Figure 4**, and g, shown in **Figure 5**, to evaluate the expressions.

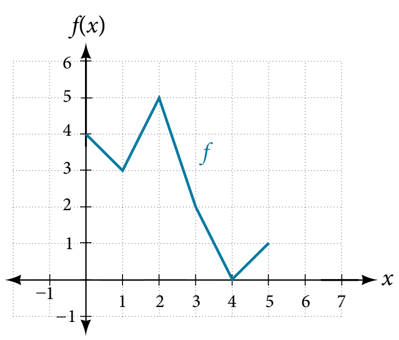


Figure 4

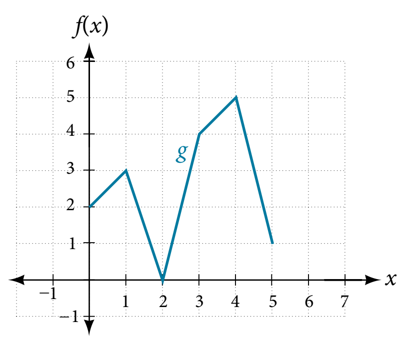


Figure 5

42. $f(g(3))$

43. $f(g(1))$

44. $g(f(1))$

45. $g(f(0))$

46. $f(f(5))$

47. $f(f(4))$

48. $g(g(2))$

49. $g(g(0))$

For the following exercises, use graphs of $f(x)$, shown in **Figure 6**, $g(x)$, shown in **Figure 7**, and $h(x)$, shown in **Figure 8**, to evaluate the expressions.

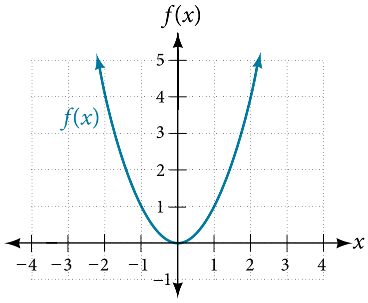


Figure 6

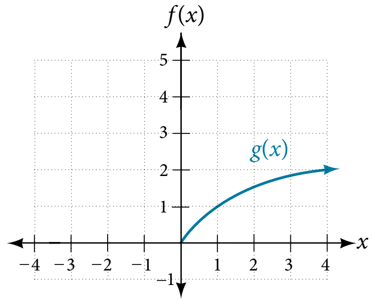


Figure 7

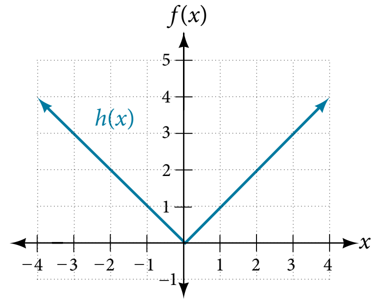


Figure 8

50. $g(f(1))$ **51.** $g(f(2))$ **52.** $f(g(4))$ **53.** $f(g(1))$

54. $f(h(2))$ **55.** $h(f(2))$ **56.** $f(g(h(4)))$ **57.** $f(g(f(-2)))$

NUMERIC

For the following exercises, use the function values for f and g shown in **Table 3** to evaluate each expression.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	7	6	5	8	4	0	2	1	9	3
$g(x)$	9	5	6	2	1	8	7	3	4	0

Table 3

58. $f(g(8))$ **59.** $f(g(5))$ **60.** $g(f(5))$ **61.** $g(f(3))$

62. $f(f(4))$ **63.** $f(f(1))$ **64.** $g(g(2))$ **65.** $g(g(6))$

For the following exercises, use the function values for f and g shown in **Table 4** to evaluate the expressions.

x	−3	−2	−1	0	1	2	3
$f(x)$	11	9	7	5	3	1	−1
$g(x)$	−8	−3	0	1	0	−3	−8

Table 4

66. $(f \circ g)(1)$ **67.** $(f \circ g)(2)$ **68.** $(g \circ f)(2)$

69. $(g \circ f)(3)$ **70.** $(g \circ g)(1)$ **71.** $(f \circ f)(3)$

For the following exercises, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

72. $f(x) = 4x + 8,\ g(x) = 7 - x^2$ **73.** $f(x) = 5x + 7,\ g(x) = 4 - 2x^2$

74. $f(x) = \sqrt{x + 4},\ g(x) = 12 - x^3$ **75.** $f(x) = \dfrac{1}{x + 2},\ g(x) = 4x + 3$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

76. $f(g(2))$ **77.** $f(g(x))$ **78.** $g(f(-3))$ **79.** $(g \circ g)(x)$

EXTENSIONS

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x - 1}$.

80. Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.

81. Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

82. What is the domain of $(g \circ f)(x)$?

83. What is the domain of $(f \circ g)(x)$?

84. Let $f(x) = \dfrac{1}{x}$.

a. Find $(f \circ f)(x)$.

b. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x + 1)^5$, $f(x) = x^5$, and $g(x) = x + 1$.

85. True or False: $(g \circ f)(x) = F(x)$.

86. True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x - 2}$.

87. $(f \circ g)(6)$; $(g \circ f)(6)$

88. $(g \circ f)(a)$; $(f \circ g)(a)$

89. $(f \circ g)(11)$; $(g \circ f)(11)$

REAL-WORLD APPLICATIONS

90. The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is \$6, you would do which of the following?

a. Evaluate $D(C(6))$.

b. Evaluate $C(D(6))$.

c. Solve $D(C(x)) = 6$.

d. Solve $C(D(p)) = 6$.

91. The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain- reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?

a. Evaluate $A(m(4))$.

b. Evaluate $m(A(4))$.

c. Solve $A(m(t)) = 4$.

d. Solve $m(A(d)) = 4$.

92. A store offers customers a 30 % discount on the price x of selected items. Then, the store takes off an additional 15 % at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)

93. A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t + 2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

94. A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).

95. Use the function you found in the previous exercise to find the total area burned after 5 minutes.

96. The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\dfrac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.

a. Find the composite function $r(V(t))$.

b. Find the *exact* time when the radius reaches 10 inches.

97. The number of bacteria in a refrigerated food product is given by

$$N(T) = 23T^2 - 56T + 1,\ 3 < T < 33,$$

where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.

a. Find the composite function $N(T(t))$.

b. Find the time (round to two decimal places) when the bacteria count reaches 6,752.

LEARNING OBJECTIVES

In this section, you will:

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the *x*-axis and the *y*-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

3.5 TRANSFORMATION OF FUNCTIONS

Figure 1 (credit: "Misko"/Flickr)

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

Graphing Functions Using Vertical and Horizontal Shifts

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a **vertical shift**, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See **Figure 2** for an example.

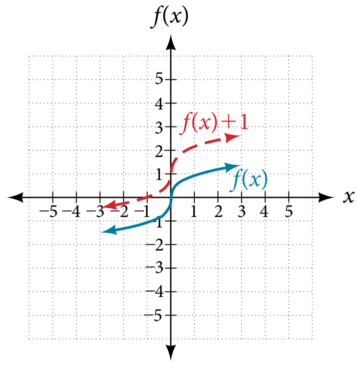


Figure 2 Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

> ***vertical shift***
> Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 1 **Adding a Constant to a Function**

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. **Figure 3** shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

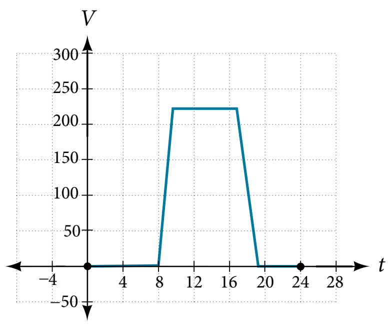


Figure 3

Solution We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in **Figure 4**.

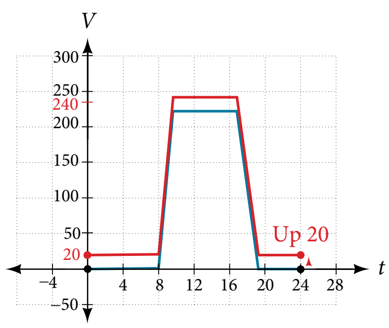


Figure 4

Notice that in **Figure 4**, for each input value, the output value has increased by 20, so if we call the new function $S(t)$, we could write

$$S(t) = V(t) + 20$$

This notation tells us that, for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See **Table 1**.

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

Table 1

How To...

Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 2 Shifting a Tabular Function Vertically

A function $f(x)$ is given in **Table 2**. Create a table for the function $g(x) = f(x) - 3$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 2

Solution The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$f(2) = 1 \quad \text{Given}$$
$$g(x) = f(x) - 3 \quad \text{Given transformation}$$
$$g(2) = f(2) - 3$$
$$= 1 - 3$$
$$= -2$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in **Table 3**.

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	−2	0	4	8

Table 3

Analysis *As with the earlier vertical shift, notice the input values stay the same and only the output values change.*

Try It #1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in **Figure 5**.

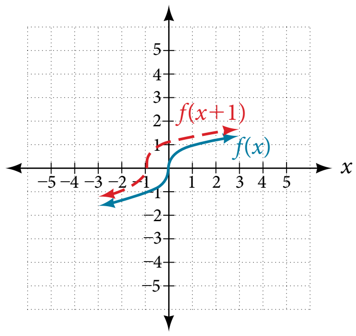


Figure 5 Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = +1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x - 2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

> ***horizontal shift***
> Given a function f, a new function $g(x) = f(x - h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 3 Adding a Constant to an Input

Returning to our building airflow example from **Figure 3**, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{the original venting plan}$$

$$F(t) = \text{starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See **Figure 6**. Notice also that the vents first opened to 220 ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220 ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t - (-2)) = V(t + 2)$.

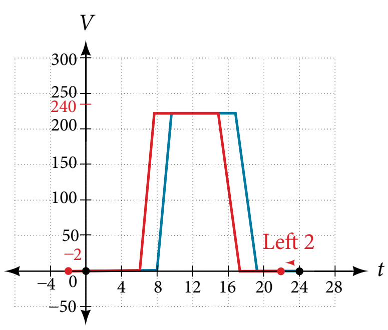


Figure 6

Analysis *Note that $V(t + 2)$ has the effect of shifting the graph to the* left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F : F(t) = V(t + 2)$.

How To...

Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 4 Shifting a Tabular Function Horizontally

A function $f(x)$ is given in **Table 4**. Create a table for the function $g(x) = f(x - 3)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 4

Solution The formula $g(x) = f(x - 3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 *larger*. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5 - 3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create **Table 5**.

x	5	7	9	11
$x - 3$	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	1	3	7	11

Table 5

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis ***Figure 7*** *represents both of the functions. We can see the horizontal shift in each point.*

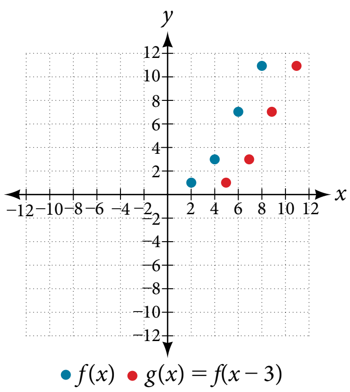


Figure 7

Example 5 Identifying a Horizontal Shift of a Toolkit Function

Figure 8 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

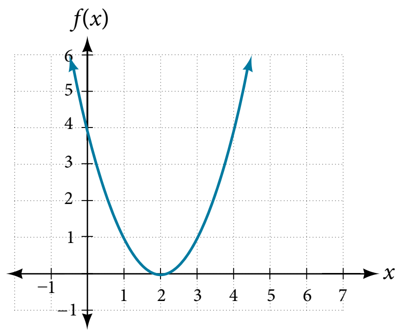


Figure 8

Solution Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at (0,0), but now the vertex is at (2,0). The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

$$f(x) = x^2$$
$$g(x) = f(x - 2)$$
$$g(x) = f(x - 2) = (x - 2)^2$$

Analysis *To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x - 2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.*

Example 6 Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m) + 10$ and $G(m + 10)$.

Solution $G(m) + 10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m + 10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Try It #2

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them. Vertical shifts are outside changes that affect the output (y-) axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input (x-) axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down *and* right or left.

How To...

Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 7 Graphing Combined Vertical and Horizontal Shifts

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x + 1) - 3$.

Solution The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x + 1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x + 1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in **Figure 9**.

Let us follow one point of the graph of $f(x) = |x|$.

- The point (0, 0) is transformed first by shifting left 1 unit: $(0, 0) \rightarrow (-1, 0)$
- The point (−1, 0) is transformed next by shifting down 3 units: $(-1, 0) \rightarrow (-1, -3)$

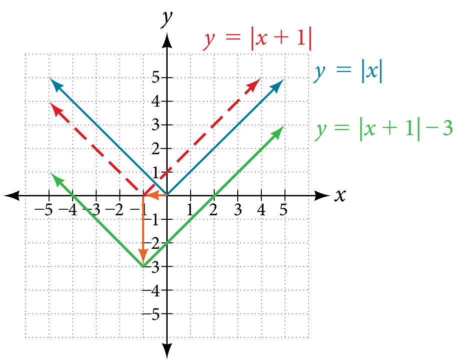


Figure 9

Figure 10 shows the graph of h.

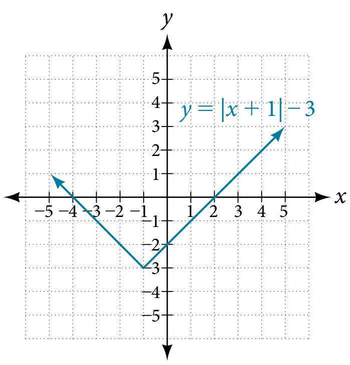


Figure 10

Try It #3

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x - 2) + 4$.

Example 8 Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in **Figure 11**, which is a transformation of the toolkit square root function.

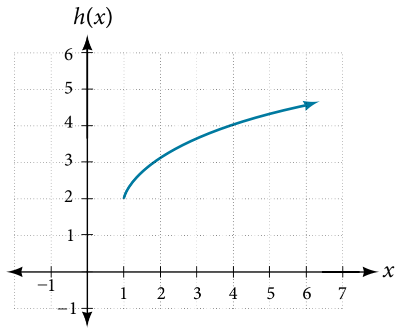


Figure 11

Solution The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x - 1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x - 1} + 2$$

Analysis *Note that this transformation has changed the domain and range of the function. This new graph has domain* $[1, \infty)$ *and range* $[2, \infty)$.

Try It #4

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in **Figure 12**.

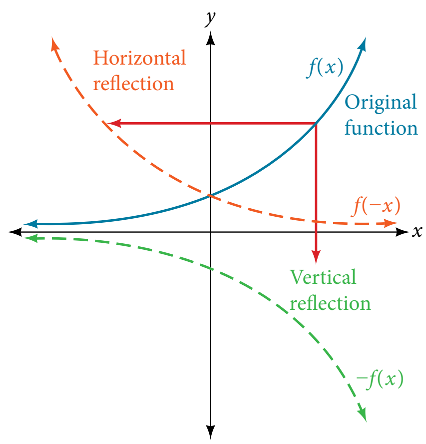


Figure 12 Vertical and horizontal reflections of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To…

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by -1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by -1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 9 Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ **a.** vertically and **b.** horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in **Figure 13**.

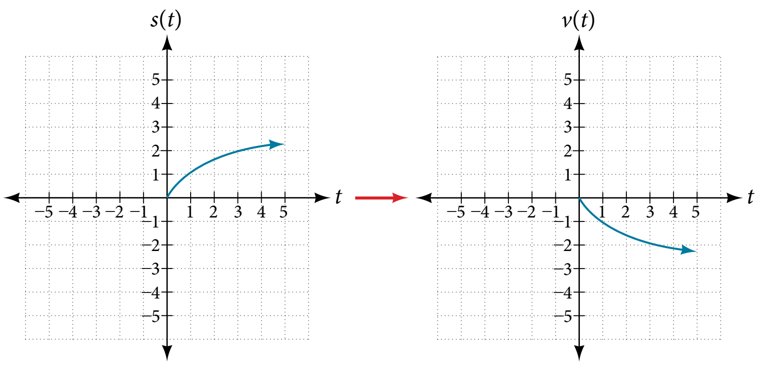


Figure 13 Vertical reflection of the square root function

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in **Figure 14**.

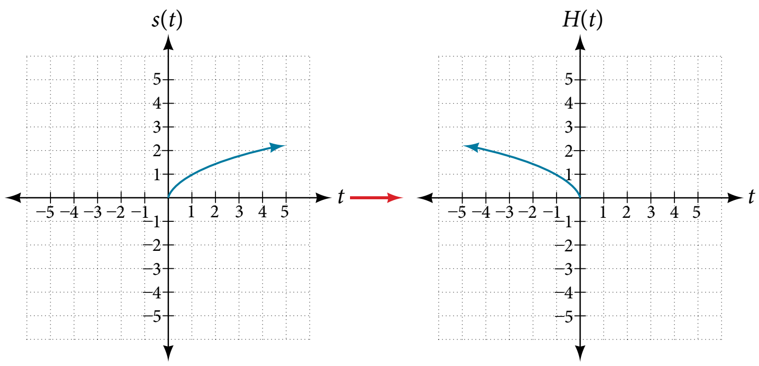


Figure 14 Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Try It #5

Reflect the graph of $f(x) = |x - 1|$ **a.** vertically and **b.** horizontally.

Example 10 Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as **Table 6**. Create a table for the functions below.

a. $g(x) = -f(x)$ **b.** $h(x) = f(-x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Table 6

Solution

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See **Table 7**.

x	2	4	6	8
$g(x)$	−1	−3	−7	−11

Table 7

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See **Table 8**.

x	−2	−4	−6	−8
$h(x)$	1	3	7	11

Table 8

Try It #6

A function $f(x)$ is given as **Table 9**. Create a table for the functions below.

x	−2	0	2	4
$f(x)$	5	10	15	20

Table 9

a. $g(x) = -f(x)$

b. $h(x) = f(-x)$

Example 11 Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in **Figure 15**. Sketch a graph of $k(t)$.

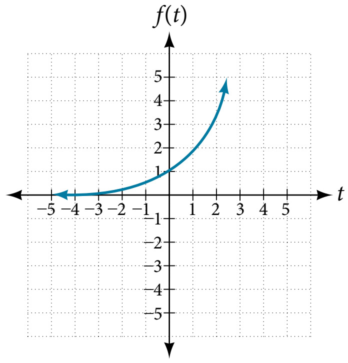


Figure 15

Solution This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points (0, 1) and (1, 2).

1. First, we apply a horizontal reflection: (0, 1) (−1, 2).
2. Then, we apply a vertical reflection: (0, −1) (1, −2).
3. Finally, we apply a vertical shift: (0, 0) (1, 1).

This means that the original points, (0,1) and (1,2) become (0,0) and (1,1) after we apply the transformations.

In **Figure 16**, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

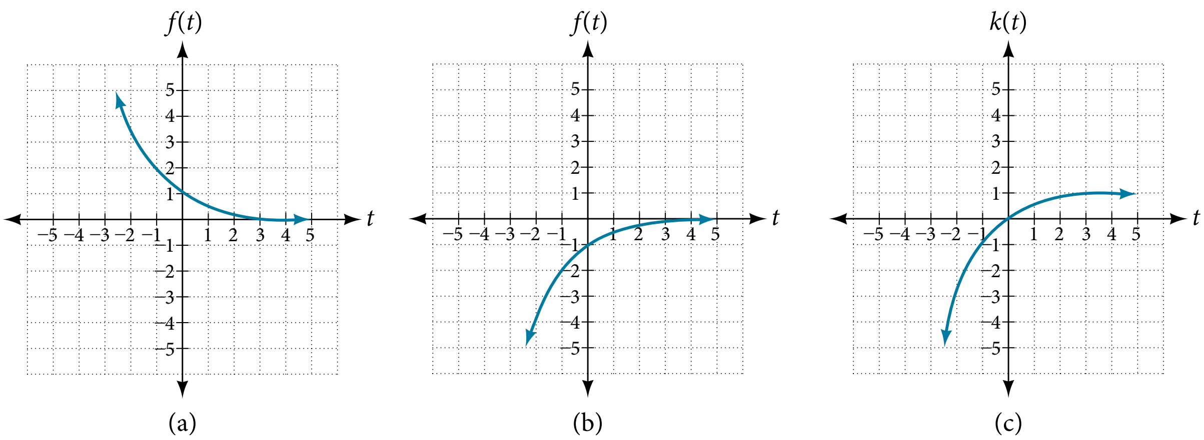


Figure 16

Analysis *As a model for learning, this function would be limited to a domain of* $t \geq 0$, *with corresponding range* [0, 1).

Try It #7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. A function whose graph is symmetric about the y-axis is called an **even function**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in **Figure 17**.

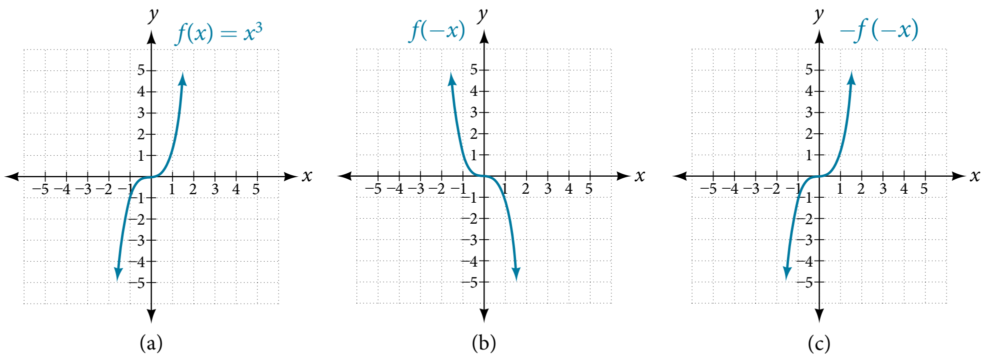


Figure 17 (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

> ***even and odd functions***
> A function is called an **even function** if for every input x: $f(x) = f(-x)$
>
> The graph of an even function is symmetric about the y-axis.
>
> A function is called an **odd function** if for every input x: $f(x) = -f(-x)$
>
> The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 12 Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis *Consider the graph of f in* ***Figure 18****. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point (−x, −y) is also on the graph. For example, (1, 3) is on the graph of f, and the corresponding point (−1, −3) is also on the graph.*

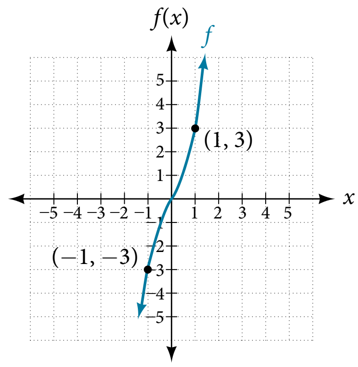


Figure 18

Try It #8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. **Figure 19** shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

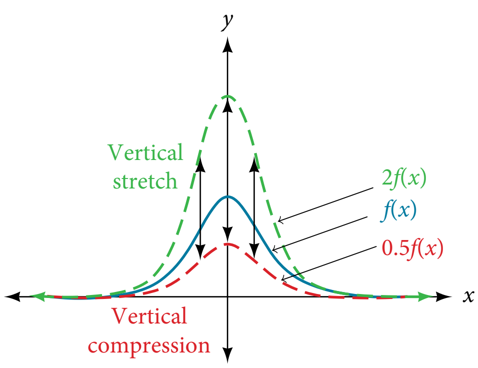


Figure 19 Vertical stretch and compression

vertical stretches and compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To...

Given a function, graph its vertical stretch.

1. Identify the value of a.
2. Multiply all range values by a.
3. If $a > 1$, the graph is stretched by a factor of a.
   If $0 < a < 1$, the graph is compressed by a factor of a.
   If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 13 **Graphing a Vertical Stretch**

A function $P(t)$ models the population of fruit flies. The graph is shown in **Figure 20**.

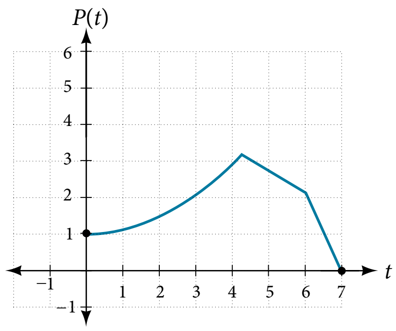


Figure 20

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in **Figure 21**.

If we choose four reference points, (0, 1), (3, 3), (6, 2) and (7, 0) we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$(0, 1) \rightarrow (0, 2)$
$(3, 3) \rightarrow (3, 6)$
$(6, 2) \rightarrow (6, 4)$
$(7, 0) \rightarrow (7, 0)$

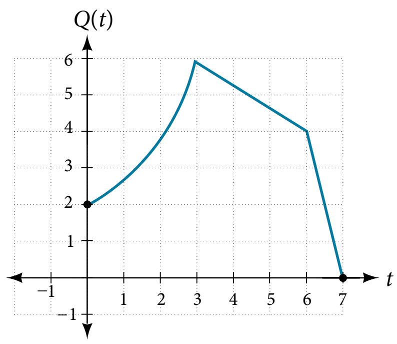


Figure 21

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To...

Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for a vertical compression.

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 14 Finding a Vertical Compression of a Tabular Function

A function f is given as **Table 10**. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 10

Solution The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce **Table 11**.

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Table 11

Analysis *The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half, so the graph is half the original height.*

Try It #9

A function f is given as **Table 12**. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

x	2	4	6	8
$f(x)$	12	16	20	0

Table 12

Example 15 Recognizing a Vertical Stretch

The graph in **Figure 22** is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

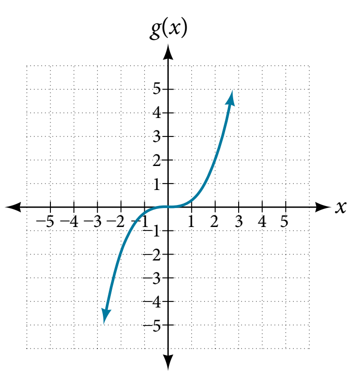


Figure 22

Solution When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$$

Try It #10

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

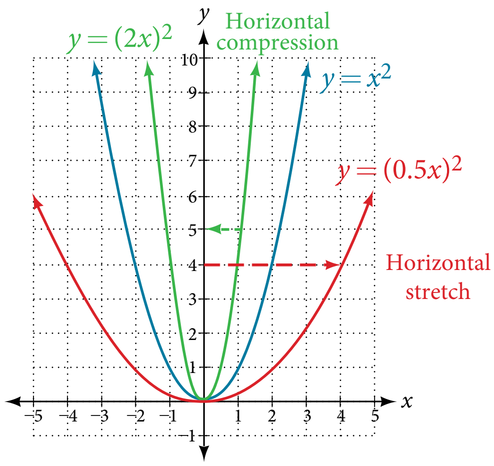


Figure 23

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe **Figure 23**. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

horizontal stretches and compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To…

Given a description of a function, sketch a horizontal compression or stretch.

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 16 Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution Symbolically, we could write

$$R(1) = P(2),$$
$$R(2) = P(4), \text{ and in general,}$$
$$R(t) = P(2t).$$

See **Figure 24** for a graphical comparison of the original population and the compressed population.

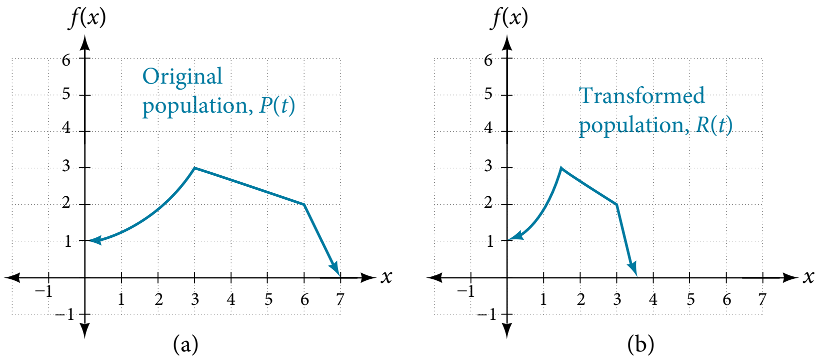


Figure 24 (a) Original population graph (b) Compressed population graph

Analysis *Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.*

Example 17 Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as **Table 13**. Create a table for the function $g(x) = f\left(\frac{1}{2}x\right)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Table 13

Solution The formula $g(x) = f\left(\frac{1}{2}x\right)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f\left(\frac{1}{2} \cdot 2\right) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f\left(\frac{1}{2} \cdot 4\right) = f(2) = 1$$

We do the same for the other values to produce **Table 14**.

x	4	8	12	16
$g(x)$	1	3	7	11

Table 14

Figure 25 shows the graphs of both of these sets of points.

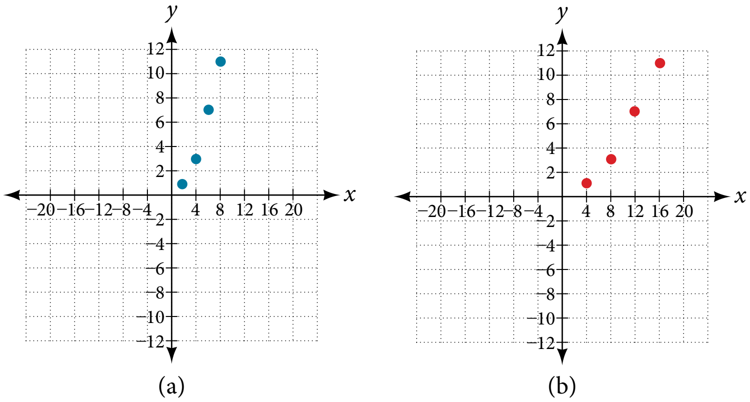


Figure 25

Analysis *Because each input value has been doubled, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.*

Example 18 Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in **Figure 26**.

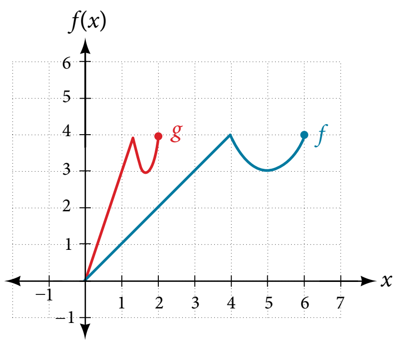


Figure 26

Solution The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at $(6, 4)$ and $g(x)$ ends at $(2, 4)$, we can see that the x-values have been compressed by $\frac{1}{3}$, because $6\left(\frac{1}{3}\right) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis *Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f\left(\frac{1}{4}x\right)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.*

Try It #11

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x) + 3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x + 3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x + 3) = 12$? We would need $2x + 3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx + p) = f\left(b\left(x + \frac{p}{b}\right)\right)$$

Let's work through an example.

$$f(x) = (2x + 4)^2$$

We can factor out a 2.

$$f(x) = (2(x + 2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

> ***combining transformations***
> When combining vertical transformations written in the form $af(x) + k$, first vertically stretch by a and then vertically shift by k.
>
> When combining horizontal transformations written in the form $f(bx - h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.
>
> When combining horizontal transformations written in the form $f(b(x - h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.
>
> Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 19 **Finding a Triple Transformation of a Tabular Function**

Given **Table 15** for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x) + 1$.

x	6	12	18	24
$f(x)$	10	14	15	17

Table 15

Solution There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$. See **Table 16**.

x	2	4	6	8
$f(3x)$	10	14	15	17

Table 16

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See **Table 17**.

x	2	4	6	8
$2f(3x)$	20	28	30	34

Table 17

Finally, we can apply the vertical shift, which will add 1 to all the output values. See **Table 18**.

x	2	4	6	8
$g(x) = 2f(3x) + 1$	21	29	31	35

Table 18

Example 20 Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in **Figure 27** to sketch a graph of $k(x) = f\left(\frac{1}{2}x + 1\right) - 3$.

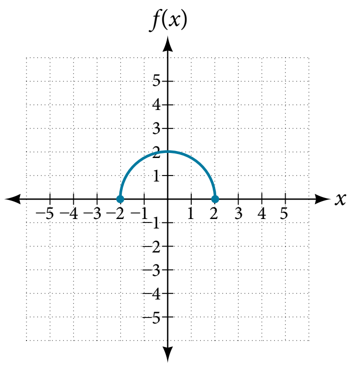


Figure 27

Solution To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x + 1\right) - 3 = f\left(\frac{1}{2}(x + 2)\right) - 3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point (0, 2) remains at (0, 2) while the point (2, 0) will stretch to (4, 0). See **Figure 28**.

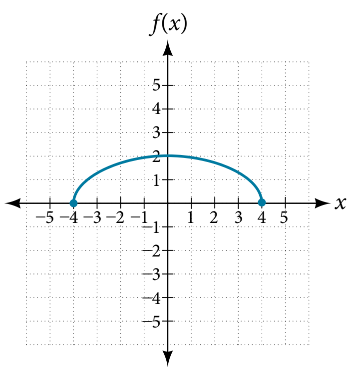


Figure 28

Next, we horizontally shift left by 2 units, as indicated by $x + 2$. See **Figure 29**.

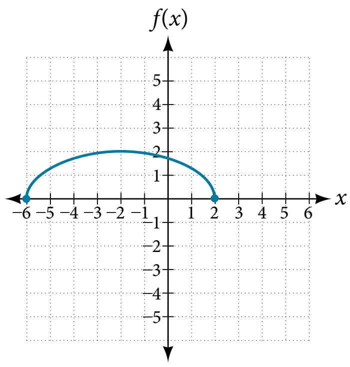


Figure 29

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See **Figure 30**.

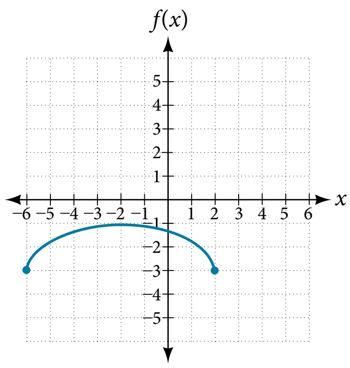


Figure 30

Access this online resource for additional instruction and practice with transformation of functions.

- Function Transformations (http://openstaxcollege.org/l/functrans)

3.5 SECTION EXERCISES

VERBAL

1. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

2. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

3. When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

4. When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

5. How can you determine whether a function is odd or even from the formula of the function?

ALGEBRAIC

For the following exercises, write a formula for the function obtained when the graph is shifted as described.

6. $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.

7. $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.

8. $f(x) = \frac{1}{x}$ is shifted down 4 units and to the right 3 units.

9. $f(x) = \frac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

10. $y = f(x - 49)$

11. $y = f(x + 43)$

12. $y = f(x + 3)$

13. $y = f(x - 4)$

14. $y = f(x) + 5$

15. $y = f(x) + 8$

16. $y = f(x) - 2$

17. $y = f(x) - 7$

18. $y = f(x - 2) + 3$

19. $y = f(x + 4) - 1$

For the following exercises, determine the interval(*s*) on which the function is increasing and decreasing.

20. $f(x) = 4(x + 1)^2 - 5$

21. $g(x) = 5(x + 3)^2 - 2$

22. $a(x) = \sqrt{-x + 4}$

23. $k(x) = -3\sqrt{x} - 1$

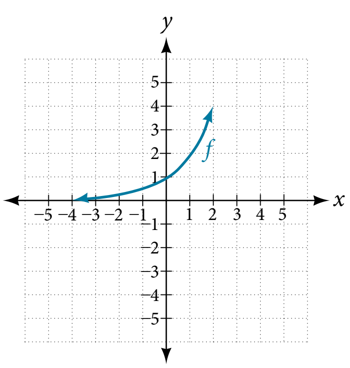


Figure 31

GRAPHICAL

For the following exercises, use the graph of $f(x) = 2^x$ shown in **Figure 31** to sketch a graph of each transformation of $f(x)$.

24. $g(x) = 2^x + 1$

25. $h(x) = 2^x - 3$

26. $w(x) = 2^{x - 1}$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

27. $f(t) = (t + 1)^2 - 3$

28. $h(x) = |x - 1| + 4$

29. $k(x) = (x - 2)^3 - 1$

30. $m(t) = 3 + \sqrt{t + 2}$

NUMERIC

31. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	–2	–1	0	1	2
$f(x)$	–2	–1	–3	1	2

x	–1	0	1	2	3
$g(x)$	–2	–1	–3	1	2

x	–2	–1	0	1	2
$h(x)$	–1	0	–2	2	3

32. Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

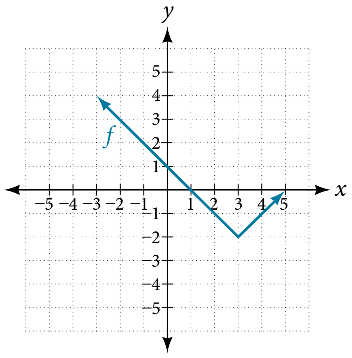
x	–2	–1	0	1	2
$f(x)$	–1	–3	4	2	1

x	–3	–2	–1	0	1
$g(x)$	–1	–3	4	2	1

x	–2	–1	0	1	2
$h(x)$	–2	–4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

33.



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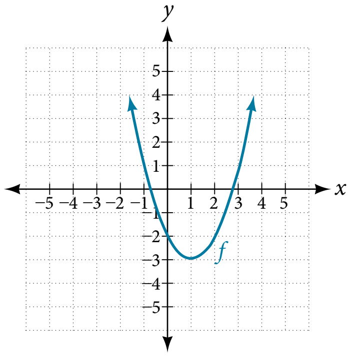

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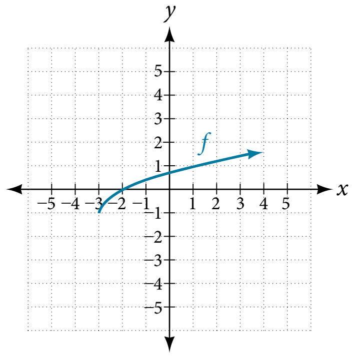

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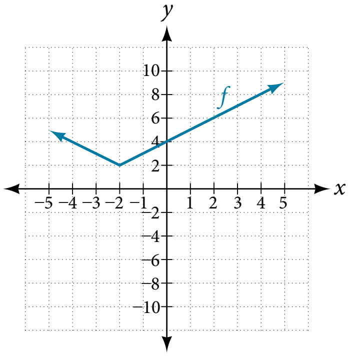

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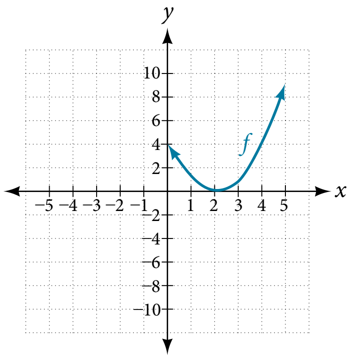

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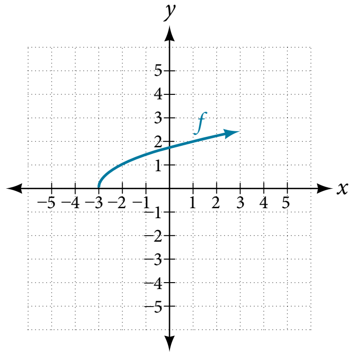

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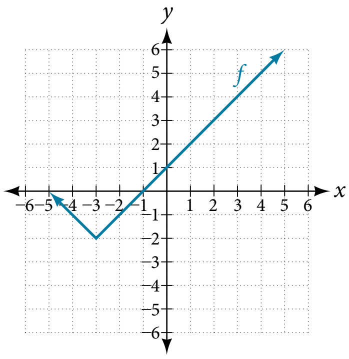

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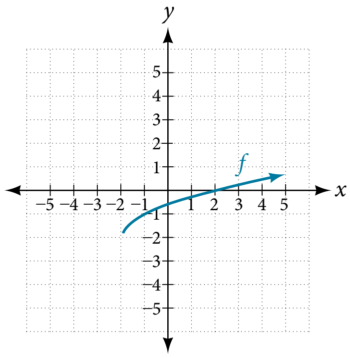
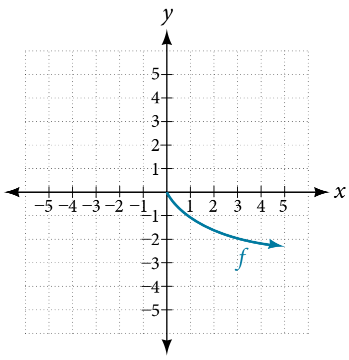
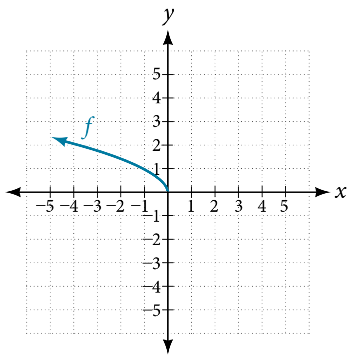


For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

41.

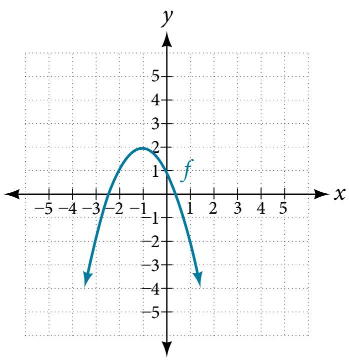


42.

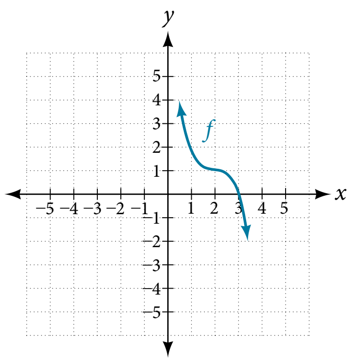


For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

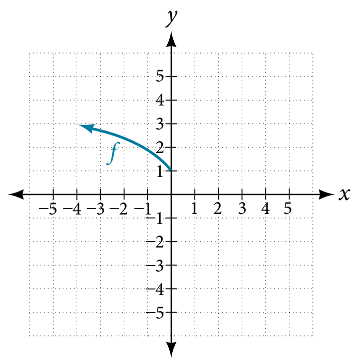
43.



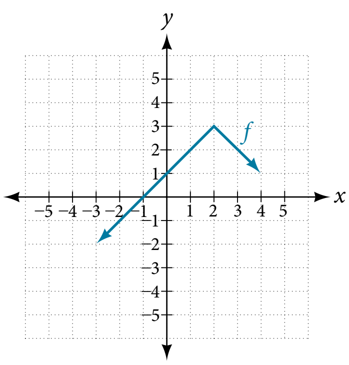
44.



45.



46.



For the following exercises, determine whether the function is odd, even, or neither.

47. $f(x) = 3x^4$

48. $g(x) = \sqrt{x}$

49. $h(x) = \frac{1}{x} + 3x$

50. $f(x) = (x - 2)^2$

51. $g(x) = 2x^4$

52. $h(x) = 2x - x^3$

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

53. $g(x) = -f(x)$

54. $g(x) = f(-x)$

55. $g(x) = 4f(x)$

56. $g(x) = 6f(x)$

57. $g(x) = f(5x)$

58. $g(x) = f(2x)$

59. $g(x) = f\left(\frac{1}{3}x\right)$

60. $g(x) = f\left(\frac{1}{5}x\right)$

61. $g(x) = 3f(-x)$

62. $g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

63. The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

64. The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.

65. The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

66. The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

67. The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

68. The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

69. $g(x) = 4(x + 1)^2 - 5$

70. $g(x) = 5(x + 3)^2 - 2$

71. $h(x) = -2|x - 4| + 3$

72. $k(x) = -3\sqrt{x} - 1$

73. $m(x) = \frac{1}{2}x^3$

74. $n(x) = \frac{1}{3}|x - 2|$

75. $p(x) = \left(\frac{1}{3}x\right)^3 - 3$

76. $q(x) = \left(\frac{1}{4}x\right)^3 + 1$

77. $a(x) = \sqrt{-x + 4}$

For the following exercises, use the graph in **Figure 32** to sketch the given transformations.

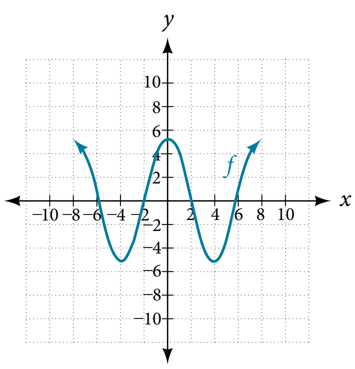


Figure 32

78. $g(x) = f(x) - 2$

79. $g(x) = -f(x)$

80. $g(x) = f(x + 1)$

81. $g(x) = f(x - 2)$

LEARNING OBJECTIVES

In this section you will:

- Graph an absolute value function.
- Solve an absolute value equation.

3.6 ABSOLUTE VALUE FUNCTIONS

Figure 1 Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will continue our investigation of absolute value functions.

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the absolute value function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign. Knowing this, we can use absolute value functions to solve some kinds of real-world problems.

absolute value function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 1 Using Absolute Value to Determine Resistance

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often $\pm 1\%$, $\pm 5\%$, or $\pm 10\%$.

Suppose we have a resistor rated at 680 ohms, $\pm 5\%$. Use the absolute value function to express the range of possible values of the actual resistance.

Solution We can find that 5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R - 680| \leq 34$$

Try It #1

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in **Figure 2**.

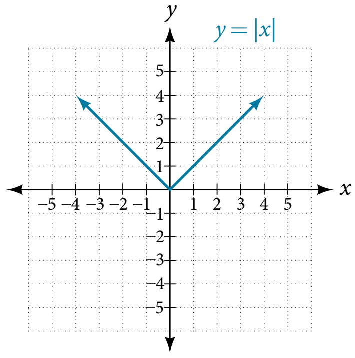


Figure 2

Figure 3 shows the graph of $y = 2|x - 3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at (3, 4) for this transformed function.

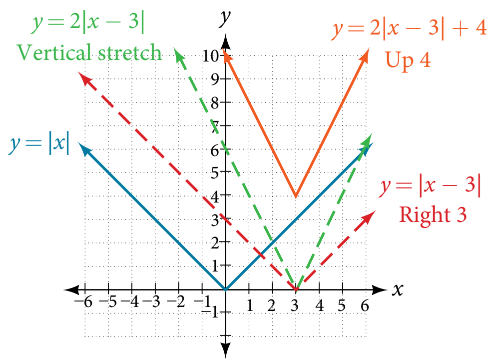


Figure 3

Example 2 Writing an Equation for an Absolute Value Function Given a Graph

Write an equation for the function graphed in **Figure 4**.

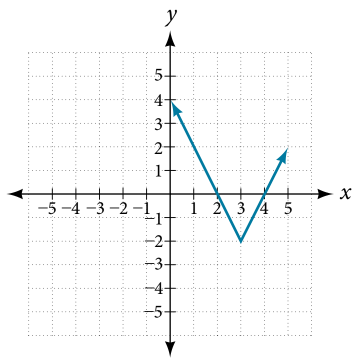


Figure 4

Solution The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See **Figure 5**.

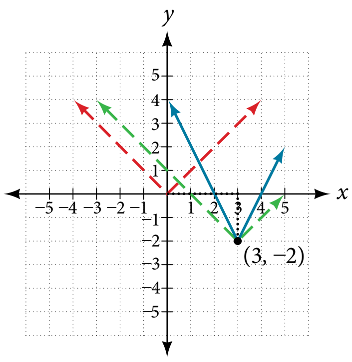


Figure 5

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in **Figure 6**.

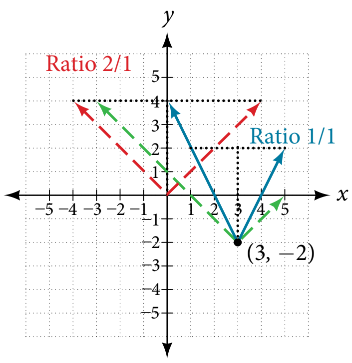


Figure 6

From this information we can write the equation

$f(x) = 2|x - 3| - 2$, treating the stretch as a vertical stretch, or

$f(x) = |2(x - 3)| - 2$, treating the stretch as a horizontal compression.

Analysis *Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.*

Q & A…

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x - 3| - 2$$

Now substituting in the point (1, 2)

$$2 = a|1 - 3| - 2$$
$$4 = 2a$$
$$a = 2$$

Try It #2

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Q & A...

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see **Figure 7**).

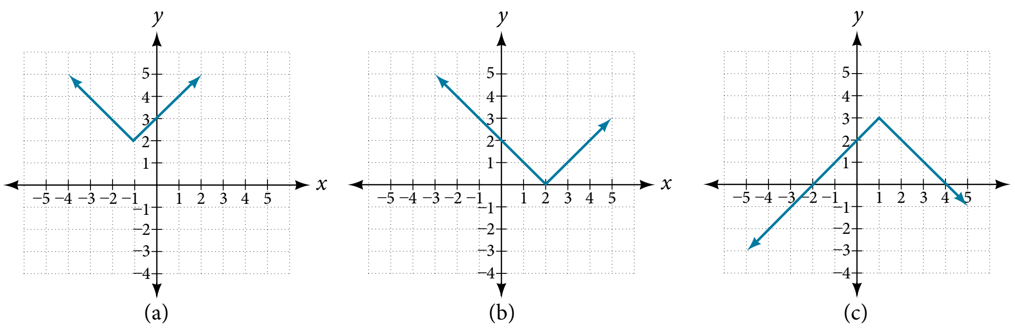


Figure 7 (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

In **Other Types of Equations**, we touched on the concepts of absolute value equations. Now that we understand a little more about their graphs, we can take another look at these types of equations. Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x - 6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$\begin{array}{lll} 2x - 6 = 8 & \text{or} & 2x - 6 = -8 \\ 2x = 14 & & 2x = -2 \\ x = 7 & & x = -1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An absolute value equation is an equation in which the unknown variable appears in absolute value bars. For example,

$$|x| = 4,$$
$$|2x - 1| = 3$$
$$|5x + 2| - 4 = 9$$

solutions to absolute value equations

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

How To...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for x.

Example 3 **Finding the Zeros of an Absolute Value Function**

For the function $f(x) = |4x + 1| - 7$, find the values of x such that $f(x) = 0$.

Solution

$0 = \|4x + 1\| - 7$	Substitute 0 for $f(x)$.
$7 = \|4x + 1\|$	Isolate the absolute value on one side of the equation.
$7 = 4x + 1$ or $-7 = 4x + 1$	Break into two separate equations and solve.
$6 = 4x \qquad -8 = 4x$	
$x = \frac{6}{4} = 1.5 \qquad x = \frac{-8}{4} = -2$	

The function outputs 0 when $x = 1.5$ or $x = -2$. See **Figure 8**.

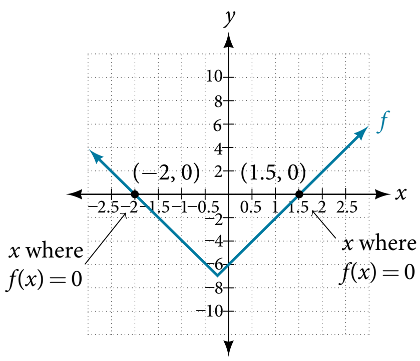


Figure 8

Try It #3

For the function $f(x) = |2x - 1| - 3$, find the values of x such that $f(x) = 0$.

Q & A...

Should we always expect two answers when solving $|A| = B$?

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

Access these online resources for additional instruction and practice with absolute value.

- Graphing Absolute Value Functions (http://openstaxcollege.org/l/graphabsvalue)
- Graphing Absolute Value Functions 2 (http://openstaxcollege.org/l/graphabsvalue2)

3.6 SECTION EXERCISES

VERBAL

1. How do you solve an absolute value equation?

2. How can you tell whether an absolute value function has two x-intercepts without graphing the function?

3. When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

4. How can you use the graph of an absolute value function to determine the x-values for which the function values are negative?

ALGEBRAIC

5. Describe all numbers x that are at a distance of 4 from the number 8. Express this using absolute value notation.

6. Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number -4. Express this using absolute value notation.

7. Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this using absolute value notation.

8. Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this using absolute value notation.

For the following exercises, find the x- and y-intercepts of the graphs of each function.

9. $f(x) = 4|x - 3| + 4$

10. $f(x) = -3|x - 2| - 1$

11. $f(x) = -2|x + 1| + 6$

12. $f(x) = -5|x + 2| + 15$

13. $f(x) = 2|x - 1| - 6$

14. $f(x) = |-2x + 1| - 13$

15. $f(x) = -|x - 9| + 16$

GRAPHICAL

For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

16. $y = |x - 1|$

17. $y = |x + 1|$

18. $y = |x| + 1$

For the following exercises, graph the given functions by hand.

19. $y = |x| - 2$

20. $y = -|x|$

21. $y = -|x| - 2$

22. $y = -|x - 3| - 2$

23. $f(x) = -|x - 1| - 2$

24. $f(x) = -|x + 3| + 4$

25. $f(x) = 2|x + 3| + 1$

26. $f(x) = 3|x - 2| + 3$

27. $f(x) = |2x - 4| - 3$

28. $f(x) = |3x + 9| + 2$

29. $f(x) = -|x - 1| - 3$

30. $f(x) = -|x + 4| -3$

31. $f(x) = \frac{1}{2}|x + 4| - 3$

TECHNOLOGY

32. Use a graphing utility to graph $f(x) = 10|x - 2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.

33. Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

For the following exercises, graph each function using a graphing utility. Specify the viewing window.

34. $f(x) = -0.1|0.1(0.2 - x)| + 0.3$

35. $f(x) = 4 \times 10^9 |x - (5 \times 10^9)| + 2 \times 10^9$

EXTENSIONS

For the following exercises, solve the inequality.

36. If possible, find all values of a such that there are no x-intercepts for $f(x) = 2|x + 1| + a$.

37. If possible, find all values of a such that there are no y-intercepts for $f(x) = 2|x + 1| + a$.

REAL-WORLD APPLICATIONS

38. Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.

39. The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.

40. Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.

41. A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.

42. The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

LEARNING OBJECTIVES

In this section, you will:

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

3.7 INVERSE FUNCTIONS

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. **Figure 1** provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

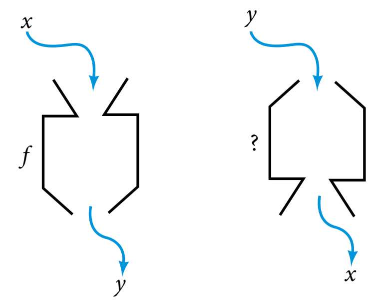


Figure 1 Can a function "machine" operate in reverse?

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24°\text{C}.$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from **Figure 2** for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

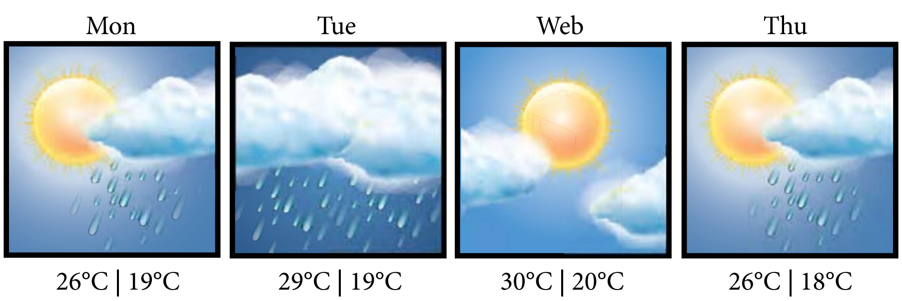


Figure 2

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32)$$

$$26 \cdot \frac{9}{5} = F - 32$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1. In other words, $f^{-1}(x)$ does *not* mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

inverse function

For any one-to-one function $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

Example 1 Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution The inverse function reverses the input and output quantities, so if

$$f(2) = 4, \text{ then } f^{-1}(4) = 2;$$
$$f(5) = 12, \text{ then } f^{-1}(12) = 5.$$

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis *Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See* ***Table 1***.

$(x, f(x))$	$(x, g(x))$
(2, 4)	(4, 2)
(5, 12)	(12, 5)

Table 1

Try It #1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

How To...

Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x)) = x$ or $g(f(x)) = x$.
2. If either statement is true, then both are true, and $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example 2 Testing Inverse Relationships Algebraically

If $f(x) = \dfrac{1}{x+2}$ and $g(x) = \dfrac{1}{x} - 2$, is $g = f^{-1}$?

Solution

$$g(f(x)) = \frac{1}{\left(\frac{1}{x+2}\right)} - 2$$
$$= x + 2 - 2$$
$$= x$$

so

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$f(g(x)) = \frac{1}{\frac{1}{x} - 2 + 2}$$
$$= \frac{1}{\frac{1}{x}}$$
$$= x$$

Analysis *Notice the inverse operations are in reverse order of the operations from the original function.*

Try It #2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x-4}$, is $g = f^{-1}$?

Example 3 Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution $f(g(x)) = \frac{x^3}{27} \neq x$

No, the functions are not inverses.

Analysis *The correct inverse to the cube is, of course, the cube root* $\sqrt[3]{x} = x^{1/3}$ *that is, the one-third is an exponent, not a multiplier.*

Try It #3

If $f(x) = (x - 1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in **Figure 3**.

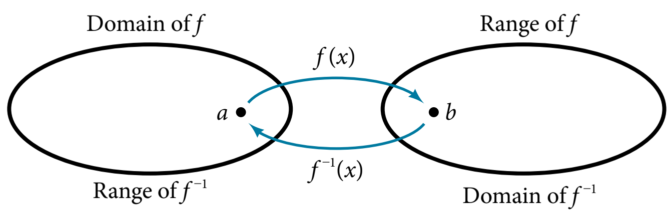


Figure 3 Domain and range of a function and its inverse

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and -3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its domain limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x - 1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of $f^{-1} = [1, \infty)$.
- The domain of f^{-1} = range of $f = [0, \infty)$.

Q & A...

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

domain and range of inverse functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$. The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To...

Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 4 Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in **Table 2**. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$

Reciprocal squared	Cube root	Square root	Absolute value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Table 2

Solution The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis *We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in* ***Figure 4****. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.*

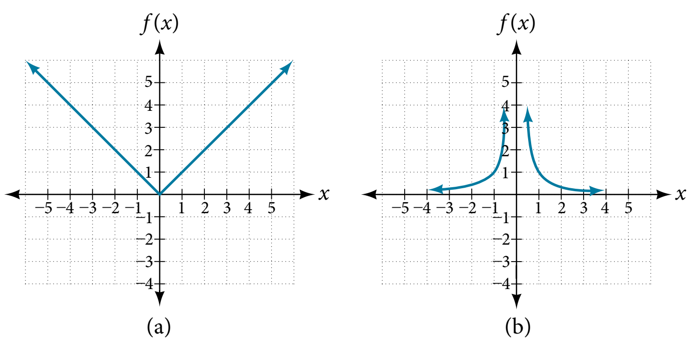


Figure 4 (a) Absolute value (b) Reciprocal squared

Try It #4

The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 5 **Interpreting the Inverse of a Tabular Function**

A function $f(t)$ is given in **Table 3**, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$.

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Table 3

Solution The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Try It #5

Using **Table 4**, find and interpret **a.** $f(60)$, and **b.** $f^{-1}(60)$.

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Table 4

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

How To...

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 6 **Evaluating a Function and Its Inverse from a Graph at Specific Points**

A function $g(x)$ is given in **Figure 5**. Find $g(3)$ and $g^{-1}(3)$.

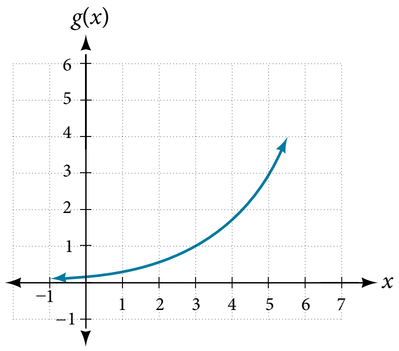


Figure 5

Solution To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point (3, 1) tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point (5, 3) on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See **Figure 6**.

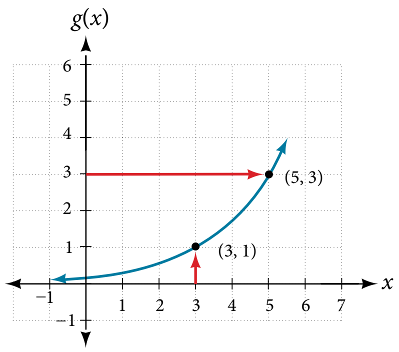


Figure 6

Try It #6

Using the graph in **Figure 6**, **a.** find $g^{-1}(1)$, and **b.** estimate $g^{-1}(4)$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula—for example, y as a function of x—we can often find the inverse function by solving to obtain x as a function of y.

How To…

Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x.
3. Interchange x and y.

Example 7 **Inverting the Fahrenheit-to-Celsius Function**

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32)$$

Solution

$$C = \frac{5}{9}(F - 32)$$
$$C \cdot \frac{9}{5} = F - 32$$
$$F = \frac{9}{5}C + 32$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32),$$

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Try It #7

Solve for x in terms of y given $y = \frac{1}{3}(x - 5)$

Example 8 **Solving to Find an Inverse Function**

Find the inverse of the function $f(x) = \frac{2}{x - 3} + 4$.

Solution

$$y = \frac{2}{x - 3} + 4 \quad \text{Set up an equation.}$$
$$y - 4 = \frac{2}{x - 3} \quad \text{Subtract 4 from both sides.}$$
$$x - 3 = \frac{2}{y - 4} \quad \text{Multiply both sides by } x - 3 \text{ and divide by } y - 4.$$
$$x = \frac{2}{y - 4} + 3 \quad \text{Add 3 to both sides.}$$

So $f^{-1}(y) = \frac{2}{y - 4} + 3$ or $f^{-1}(x) = \frac{2}{x - 4} + 3$.

Analysis *The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating* ***Table 5***.

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Table 5

Example 9 **Solving to Find an Inverse with Radicals**

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

Solution

$$y = 2 + \sqrt{x - 4}$$
$$(y - 2)^2 = x - 4$$
$$x = (y - 2)^2 + 4$$

So $f^{-1}(x) = (x - 2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

Analysis *The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.*

Try It #8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in **Figure 7**.

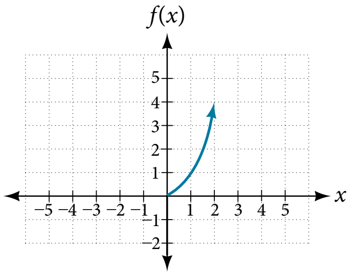


Figure 7 Quadratic function with domain restricted to [0, ∞).

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in **Figure 8**.

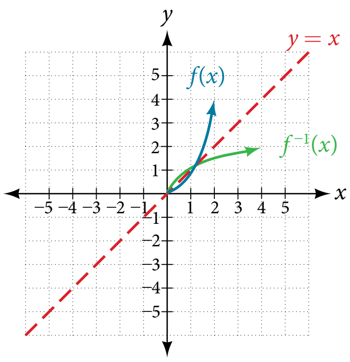


Figure 8 Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 10 Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in **Figure 9**, sketch a graph of $f^{-1}(x)$.

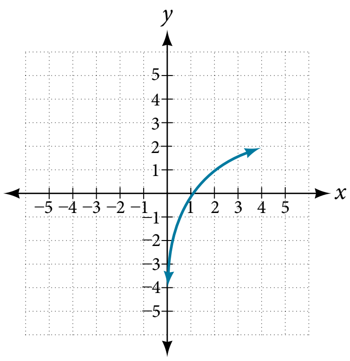


Figure 9

Solution This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives **Figure 10**.

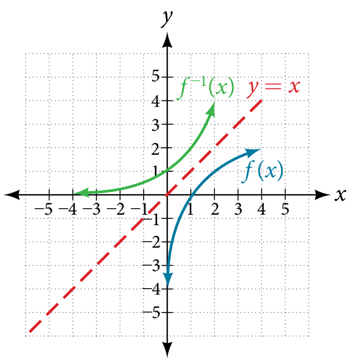


Figure 10 The function and its inverse, showing reflection about the identity line

Try It #9

Draw graphs of the functions f and f^{-1} from **Example 8**.

Q & A...

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Access these online resources for additional instruction and practice with inverse functions.

- Inverse Functions (http://openstaxcollege.org/l/inversefunction)
- One-to-one Functions (http://openstaxcollege.org/l/onetoone)
- Inverse Function Values Using Graph (http://openstaxcollege.org/l/inversfuncgraph)
- Restricting the Domain and Finding the Inverse (http://openstaxcollege.org/l/restrictdomain)

3.7 SECTION EXERCISES

VERBAL

1. Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

2. Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

3. Can a function be its own inverse? Explain.

4. Are one-to-one functions either always increasing or always decreasing? Why or why not?

5. How do you find the inverse of a function algebraically?

ALGEBRAIC

6. Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

7. $f(x) = x + 3$
8. $f(x) = x + 5$
9. $f(x) = 2 - x$
10. $f(x) = 3 - x$
11. $f(x) = \dfrac{x}{x+2}$
12. $f(x) = \dfrac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

13. $f(x) = (x+7)^2$
14. $f(x) = (x-6)^2$
15. $f(x) = x^2 - 5$

16. Given $f(x) = \dfrac{x}{2+x}$ and $g(x) = \dfrac{2x}{1-x}$:
   a. Find $f(g(x))$ and $g(f(x))$.
   b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

17. $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$
18. $f(x) = -3x + 5$ and $g(x) = \dfrac{x-5}{-3}$

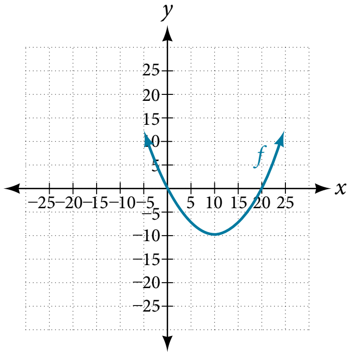
GRAPHICAL

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

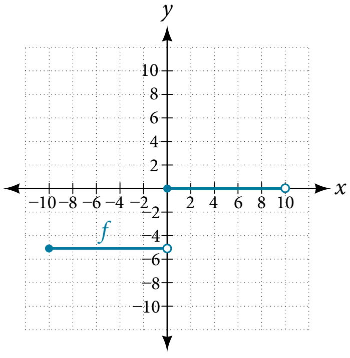
19. $f(x) = \sqrt{x}$
20. $f(x) = \sqrt[3]{3x+1}$
21. $f(x) = -5x + 1$
22. $f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

23.



24.



For the following exercises, use the graph of f shown in **Figure 11**.

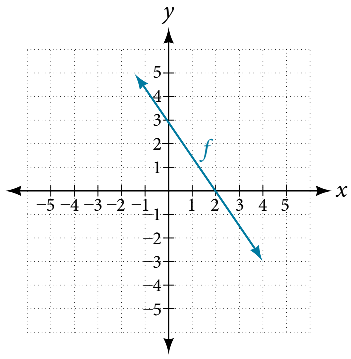


Figure 11

25. Find $f(0)$.

26. Solve $f(x) = 0$.

27. Find $f^{-1}(0)$.

28. Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in **Figure 12**.

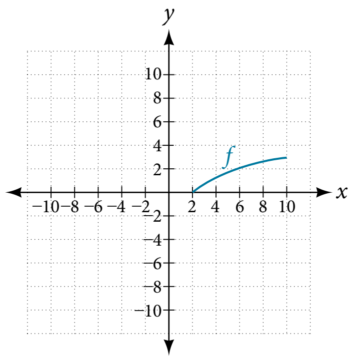


Figure 12

29. Sketch the graph of f^{-1}.

30. Find $f(6)$ and $f^{-1}(2)$.

31. If the complete graph of f is shown, find the domain of f.

32. If the complete graph of f is shown, find the range of f.

NUMERIC

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

33. If $f(6) = 7$, find $f^{-1}(7)$.

34. If $f(3) = 2$, find $f^{-1}(2)$.

35. If $f^{-1}(-4) = -8$, find $f(-8)$.

36. If $f^{-1}(-2) = -1$, find $f(-1)$.

For the following exercises, use the values listed in **Table 6** to evaluate or solve.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

Table 6

37. Find $f(1)$.

38. Solve $f(x) = 3$.

39. Find $f^{-1}(0)$.

40. Solve $f^{-1}(x) = 7$.

41. Use the tabular representation of f in **Table 7** to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Table 7

TECHNOLOGY

For the following exercises, find the inverse function. Then, graph the function and its inverse.

42. $f(x) = \dfrac{3}{x-2}$

43. $f(x) = x^3 - 1$

44. Find the inverse function of $f(x) = \dfrac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

REAL-WORLD APPLICATIONS

45. To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \dfrac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

46. The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

47. A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

CHAPTER 3 REVIEW

Key Terms

absolute maximum the greatest value of a function over an interval

absolute minimum the lowest value of a function over an interval

average rate of change the difference in the output values of a function found for two values of the input divided by the difference between the inputs

composite function the new function formed by function composition, when the output of one function is used as the input of another

decreasing function a function is decreasing in some open interval if $f(b) < f(a)$ for any two input values a and b in the given interval where $b > a$

dependent variable an output variable

domain the set of all possible input values for a relation

even function a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

function a relation in which each input value yields a unique output value

horizontal compression a transformation that compresses a function's graph horizontally, by multiplying the input by a constant $b > 1$

horizontal line test a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

horizontal reflection a transformation that reflects a function's graph across the y-axis by multiplying the input by -1

horizontal shift a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch a transformation that stretches a function's graph horizontally by multiplying the input by a constant $0 < b < 1$

increasing function a function is increasing in some open interval if $f(b) > f(a)$ for any two input values a and b in the given interval where $b > a$

independent variable an input variable

input each object or value in a domain that relates to another object or value by a relationship known as a function

interval notation a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

inverse function for any one-to-one function $f(x)$, the inverse is a function $f^{-1}(x)$ such that $f^{-1}(f(x)) = x$ for all x in the domain of f; this also implies that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}

local extrema collectively, all of a function's local maxima and minima

local maximum a value of the input where a function changes from increasing to decreasing as the input value increases.

local minimum a value of the input where a function changes from decreasing to increasing as the input value increases.

odd function a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

one-to-one function a function for which each value of the output is associated with a unique input value

output each object or value in the range that is produced when an input value is entered into a function

piecewise function a function in which more than one formula is used to define the output

range the set of output values that result from the input values in a relation

rate of change the change of an output quantity relative to the change of the input quantity

relation a set of ordered pairs

set-builder notation a method of describing a set by a rule that all of its members obey; it takes the form $\{x \mid \text{statement about } x\}$

vertical compression a function transformation that compresses the function's graph vertically by multiplying the output by a constant $0 < a < 1$

vertical line test a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

vertical reflection a transformation that reflects a function's graph across the x-axis by multiplying the output by -1

vertical shift a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

vertical stretch a transformation that stretches a function's graph vertically by multiplying the output by a constant $a > 1$

Key Equations

Constant function	$f(x) = c$, where c is a constant
Identity function	$f(x) = x$
Absolute value function	$f(x) = \|x\|$
Quadratic function	$f(x) = x^2$
Cubic function	$f(x) = x^3$
Reciprocal function	$f(x) = \frac{1}{x}$
Reciprocal squared function	$f(x) = \frac{1}{x^2}$
Square root function	$f(x) = \sqrt{x}$
Cube root function	$f(x) = \sqrt[3]{x}$
Average rate of change	$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
Composite function	$(f \circ g)(x) = f(g(x))$
Vertical shift	$g(x) = f(x) + k$ (up for $k > 0$)
Horizontal shift	$g(x) = f(x - h)$ (right for $h > 0$)
Vertical reflection	$g(x) = -f(x)$
Horizontal reflection	$g(x) = f(-x)$
Vertical stretch	$g(x) = af(x)$ $(a > 0)$
Vertical compression	$g(x) = af(x)$ $(0 < a < 1)$
Horizontal stretch	$g(x) = f(bx)$ $(0 < b < 1)$
Horizontal compression	$g(x) = f(bx)$ $(b > 1)$

Key Concepts

3.1 Functions and Function Notation

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See **Example 1** and **Example 2**.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$. See **Example 3** and **Example 4**.
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See **Example 5**.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See **Example 6** and **Example 7**.
- To solve for a specific function value, we determine the input values that yield the specific output value. See **Example 8**.
- An algebraic form of a function can be written from an equation. See **Example 9** and **Example 10**.
- Input and output values of a function can be identified from a table. See **Example 11**.
- Relating input values to output values on a graph is another way to evaluate a function. See **Example 12**.
- A function is one-to-one if each output value corresponds to only one input value. See **Example 13**.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See **Example 14**.
- The graph of a one-to-one function passes the horizontal line test. See **Example 15**.

3.2 Domain and Range

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs. See **Example 1.**
- The domain of a function can also be determined by identifying the input values of a function written as an equation. See **Example 2**, **Example 3**, and **Example 4**.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation. See **Example 5**.
- For many functions, the domain and range can be determined from a graph. See **Example 6** and **Example 7**.
- An understanding of toolkit functions can be used to find the domain and range of related functions. See **Example 8**, **Example 9**, and **Example 10**.
- A piecewise function is described by more than one formula. See **Example 11** and **Example 12**.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain. See **Example 13**.

3.3 Rates of Change and Behavior of Graphs

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. See **Example 1**.
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See **Example 2**.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See **Example 3**.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See **Example 4** and **Example 5**.
- The average rate of change can sometimes be determined as an expression. See **Example 6**.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See **Example 7**.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.

- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See **Example 8** and **Example 9**.
- The highest and lowest points on a graph indicate the maxima and minima. See **Example 10**.

3.4 Composition of Functions

- We can perform algebraic operations on functions. See **Example 1**.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See **Example 2** and **Example 3**.
- The order of function composition must be considered when interpreting the meaning of composite functions. See **Example 4**.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See **Example 5**.
- A composite function can be evaluated from a graph. See **Example 6**.
- A composite function can be evaluated from a formula. See **Example 7**.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See **Example 8** and **Example 9**.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See **Example 10**.

3.5 Transformation of Functions

- A function can be shifted vertically by adding a constant to the output. See **Example 1** and **Example 2**.
- A function can be shifted horizontally by adding a constant to the input. See **Example 3**, **Example 4**, and **Example 5**.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts. See **Example 6**.
- Vertical and horizontal shifts are often combined. See **Example 7** and **Example 8**.
- A vertical reflection reflects a graph about the x-axis. A graph can be reflected vertically by multiplying the output by -1.
- A horizontal reflection reflects a graph about the y-axis. A graph can be reflected horizontally by multiplying the input by -1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph. See **Example 9**.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly. See **Example 10**.
- A function presented as an equation can be reflected by applying transformations one at a time. See **Example 11**.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither. See **Example 12**.
- A function can be compressed or stretched vertically by multiplying the output by a constant. See **Example 13**, **Example 14**, and **Example 15**.
- A function can be compressed or stretched horizontally by multiplying the input by a constant. See **Example 16**, **Example 17**, and **Example 18**.

- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order. See **Example 19** and **Example 20**.

3.6 Absolute Value Functions

- Applied problems, such as ranges of possible values, can also be solved using the absolute value function. See **Example 1**.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction. See **Example 2**.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable. See **Example 3**.

3.7 Inverse Functions

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$. See **Example 1**, **Example 2**, and **Example 3**.
- Each of the toolkit functions has an inverse. See **Example 4**.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse. See **Example 5**.
- The inverse of a function can be determined at specific points on its graph. See **Example 6**.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y. See **Example 7**, **Example 8**, and **Example 9**.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$. See **Example 10**.

CHAPTER 3 REVIEW EXERCISES

FUNCTIONS AND FUNCTION NOTATION

For the following exercises, determine whether the relation is a function.

1. $\{(a, b), (c, d), (e, d)\}$

2. $\{(5, 2), (6, 1), (6, 2), (4, 8)\}$

3. $y^2 + 4 = x$, for x the independent variable and y the dependent variable

4. Is the graph in **Figure 1** a function?

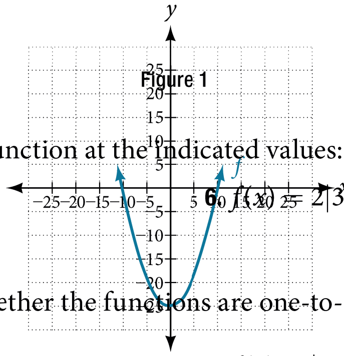


Figure 1

For the following exercises, evaluate the function at the indicated values: $f(-3); f(2); f(-a); -f(a); f(a + h)$.

5. $f(x) = -2x^2 + 3x$

6. $f(x) = 2|3x - 1|$

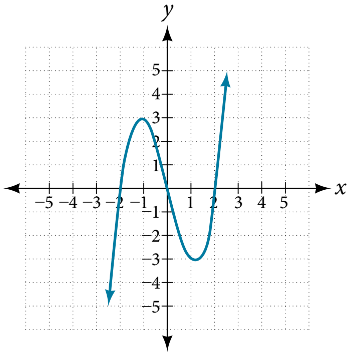
For the following exercises, determine whether the functions are one-to-one.

7. $f(x) = -3x + 5$

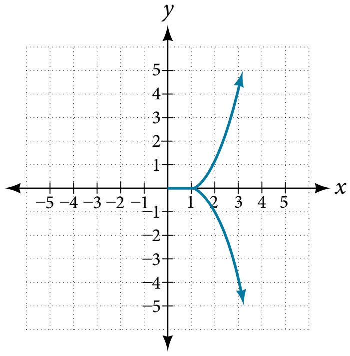
8. $f(x) = |x - 3|$

For the following exercises, use the vertical line test to determine if the relation whose graph is provided is a function.

9.



10.



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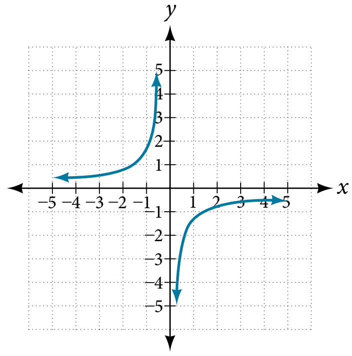


For the following exercises, graph the functions.

12. $f(x) = |x + 1|$

13. $f(x) = x^2 - 2$

For the following exercises, use **Figure 2** to approximate the values.

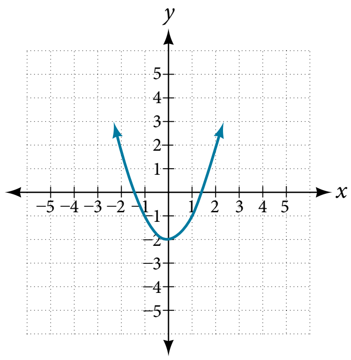


Figure 2

14. $f(2)$

15. $f(-2)$

16. If $f(x) = -2$, then solve for x.

17. If $f(x) = 1$, then solve for x.

For the following exercises, use the function $h(t) = -16t^2 + 80t$ to find the values.

18. $\dfrac{h(2) - h(1)}{2 - 1}$

19. $\dfrac{h(a) - h(1)}{a - 1}$

DOMAIN AND RANGE

For the following exercises, find the domain of each function, expressing answers using interval notation.

20. $f(x) = \dfrac{2}{3x + 2}$

21. $f(x) = \dfrac{x - 3}{x^2 - 4x - 12}$

22. $f(x) = \dfrac{\sqrt{x - 6}}{\sqrt{x - 4}}$

23. Graph this piecewise function: $f(x) = \begin{cases} x + 1 & x < -2 \\ -2x - 3 & x \geq -2 \end{cases}$

RATES OF CHANGE AND BEHAVIOR OF GRAPHS

For the following exercises, find the average rate of change of the functions from $x = 1$ to $x = 2$.

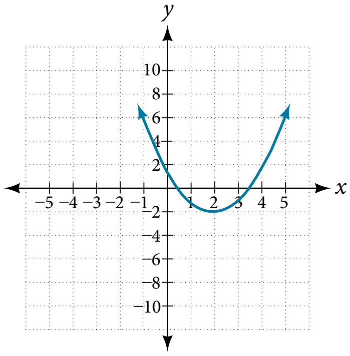
24. $f(x) = 4x - 3$

25. $f(x) = 10x^2 + x$

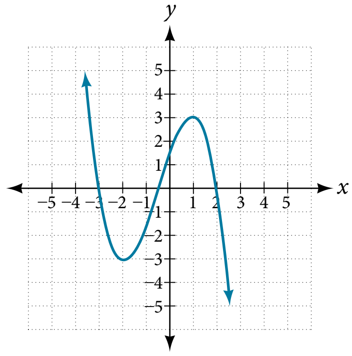
26. $f(x) = -\dfrac{2}{x^2}$

For the following exercises, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.

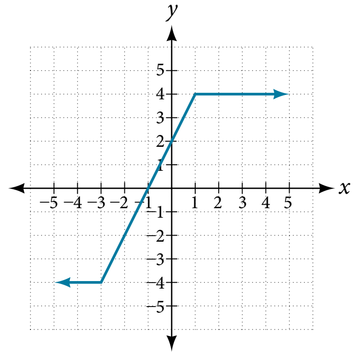
27.



28.



29.



30. Find the local minimum of the function graphed in **Exercise 27**.

31. Find the local extrema for the function graphed in **Exercise 28**.

32. For the graph in **Figure 3**, the domain of the function is $[-3, 3]$. The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

33. Find the absolute maximum of the function graphed in **Figure 3**.

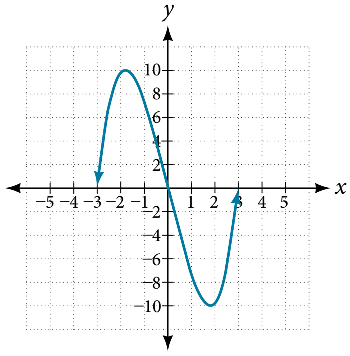


Figure 3

COMPOSITION OF FUNCTIONS

For the following exercises, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.

34. $f(x) = 4 - x,\ g(x) = -4x$

35. $f(x) = 3x + 2,\ g(x) = 5 - 6x$

36. $f(x) = x^2 + 2x,\ g(x) = 5x + 1$

37. $f(x) = \sqrt{x + 2},\ g(x) = \dfrac{1}{x}$

38. $f(x) = \dfrac{x + 3}{2},\ g(x) = \sqrt{1 - x}$

For the following exercises, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.

39. $f(x) = \dfrac{x + 1}{x + 4},\ g(x) = \dfrac{1}{x}$

40. $f(x) = \dfrac{1}{x + 3},\ g(x) = \dfrac{1}{x - 9}$

41. $f(x) = \dfrac{1}{x},\ g(x) = \sqrt{x}$

42. $f(x) = \dfrac{1}{x^2 - 1},\ g(x) = \sqrt{x + 1}$

For the following exercises, express each function H as a composition of two functions f and g where $H(x) = (f \circ g)(x)$.

43. $H(x) = \sqrt{\dfrac{2x - 1}{3x + 4}}$

44. $H(x) = \dfrac{1}{(3x^2 - 4)^{-3}}$

TRANSFORMATION OF FUNCTIONS

For the following exercises, sketch a graph of the given function.

45. $f(x) = (x - 3)^2$

46. $f(x) = (x + 4)^3$

47. $f(x) = \sqrt{x} + 5$

48. $f(x) = -x^3$

49. $f(x) = \sqrt[3]{-x}$

50. $f(x) = 5\sqrt{-x} - 4$

51. $f(x) = 4[|x - 2| - 6]$

52. $f(x) = -(x + 2)^2 - 1$

For the following exercises, sketch the graph of the function g if the graph of the function f is shown in **Figure 4**.

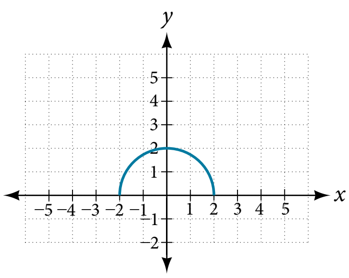


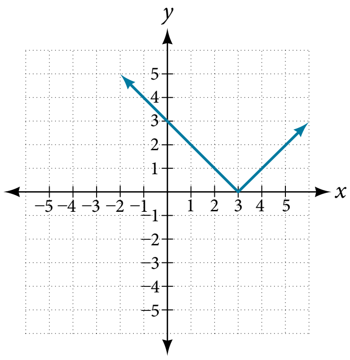
Figure 4

53. $g(x) = f(x - 1)$

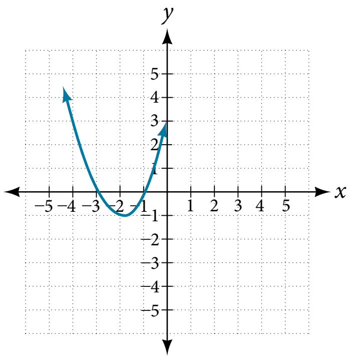
54. $g(x) = 3f(x)$

For the following exercises, write the equation for the standard function represented by each of the graphs below.

55.



56.



For the following exercises, determine whether each function below is even, odd, or neither.

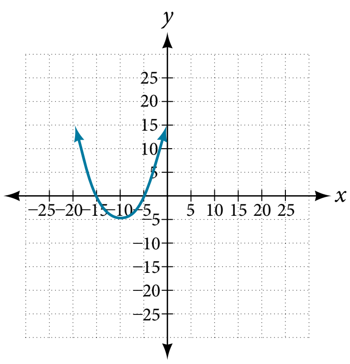
57. $f(x) = 3x^4$

58. $g(x) = \sqrt{x}$

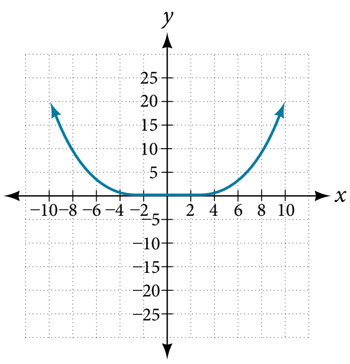
59. $h(x) = \dfrac{1}{x} + 3x$

For the following exercises, analyze the graph and determine whether the graphed function is even, odd, or neither.

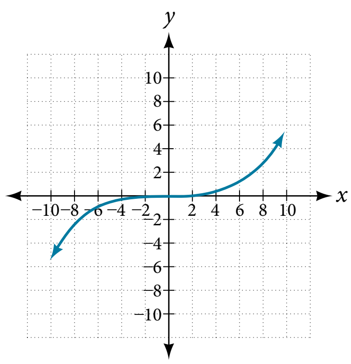
60.



61.



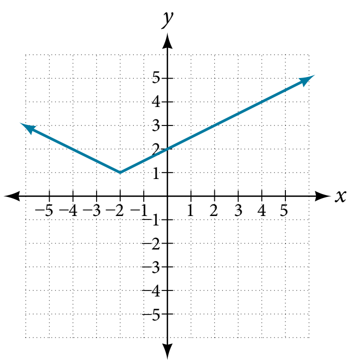
62.



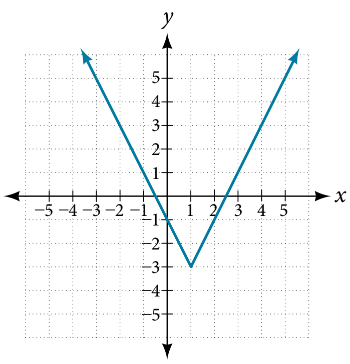
ABSOLUTE VALUE FUNCTIONS

For the following exercises, write an equation for the transformation of $f(x) = |x|$.

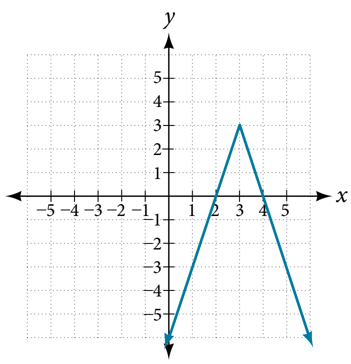
63.



64.



65.



For the following exercises, graph the absolute value function.

66. $f(x) = |x - 5|$

67. $f(x) = -|x - 3|$

68. $f(x) = |2x - 4|$

INVERSE FUNCTIONS

For the following exercises, find $f^{-1}(x)$ for each function.

69. $f(x) = 9 + 10x$

70. $f(x) = \dfrac{x}{x + 2}$

For the following exercise, find a domain on which the function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

71. $f(x) = x^2 + 1$

72. Given $f(x) = x^3 - 5$ and $g(x) = \sqrt[3]{x + 5}$:

a. Find $f(g(x))$ and $g(f(x))$.

b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

73. $f(x) = \dfrac{1}{x}$

74. $f(x) = -3x^2 + x$

75. If $f(5) = 2$, find $f^{-1}(2)$.

76. If $f(1) = 4$, find $f^{-1}(4)$.

CHAPTER 3 PRACTICE TEST

For the following exercises, determine whether each of the following relations is a function.

1. $y = 2x + 8$

2. $\{(2, 1), (3, 2), (-1, 1), (0, -2)\}$

For the following exercises, evaluate the function $f(x) = -3x^2 + 2x$ at the given input.

3. $f(-2)$

4. $f(a)$

5. Show that the function $f(x) = -2(x - 1)^2 + 3$ is not one-to-one.

6. Write the domain of the function $f(x) = \sqrt{3 - x}$ in interval notation.

7. Given $f(x) = 2x^2 - 5x$, find $f(a + 1) - f(1)$.

8. Graph the function $f(x) = \begin{cases} x + 1 \text{ if } -2 < x < 3 \\ -x \quad \text{if} \quad x \geq 3 \end{cases}$

9. Find the average rate of change of the function $f(x) = 3 - 2x^2 + x$ by finding $\dfrac{f(b) - f(a)}{b - a}$.

For the following exercises, use the functions $f(x) = 3 - 2x^2 + x$ and $g(x) = \sqrt{x}$ to find the composite functions.

10. $(g \circ f)(x)$

11. $(g \circ f)(1)$

12. Express $H(x) = \sqrt[3]{5x^2 - 3x}$ as a composition of two functions, f and g, where $(f \circ g)(x) = H(x)$.

For the following exercises, graph the functions by translating, stretching, and/or compressing a toolkit function.

13. $f(x) = \sqrt{x + 6} - 1$

14. $f(x) = \dfrac{1}{x + 2} - 1$

For the following exercises, determine whether the functions are even, odd, or neither.

15. $f(x) = -\dfrac{5}{x^2} + 9x^6$

16. $f(x) = -\dfrac{5}{x^3} + 9x^5$

17. $f(x) = \dfrac{1}{x}$

18. Graph the absolute value function $f(x) = -2|x - 1| + 3$.

For the following exercises, find the inverse of the function.

19. $f(x) = 3x - 5$

20. $f(x) = \dfrac{4}{x + 7}$

For the following exercises, use the graph of g shown in **Figure 1**.

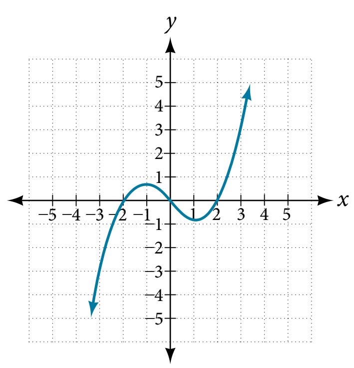


Figure 1

21. On what intervals is the function increasing?

22. On what intervals is the function decreasing?

23. Approximate the local minimum of the function. Express the answer as an ordered pair.

24. Approximate the local maximum of the function. Express the answer as an ordered pair.

For the following exercises, use the graph of the piecewise function shown in **Figure 2**.

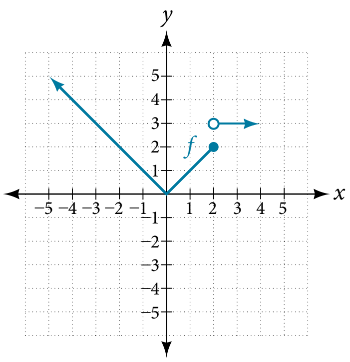


Figure 2

25. Find $f(2)$.

26. Find $f(-2)$.

27. Write an equation for the piecewise function.

For the following exercises, use the values listed in **Table 1**.

x	0	1	2	3	4	5	6	7	8
$F(x)$	1	3	5	7	9	11	13	15	17

Table 1

28. Find $F(6)$.

29. Solve the equation $F(x) = 5$.

30. Is the graph increasing or decreasing on its domain?

31. Is the function represented by the graph one-to-one?

32. Find $F^{-1}(15)$.

33. Given $f(x) = -2x + 11$, find $f^{-1}(x)$.

LEARNING OBJECTIVES

In this section, you will:

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

5.1 QUADRATIC FUNCTIONS

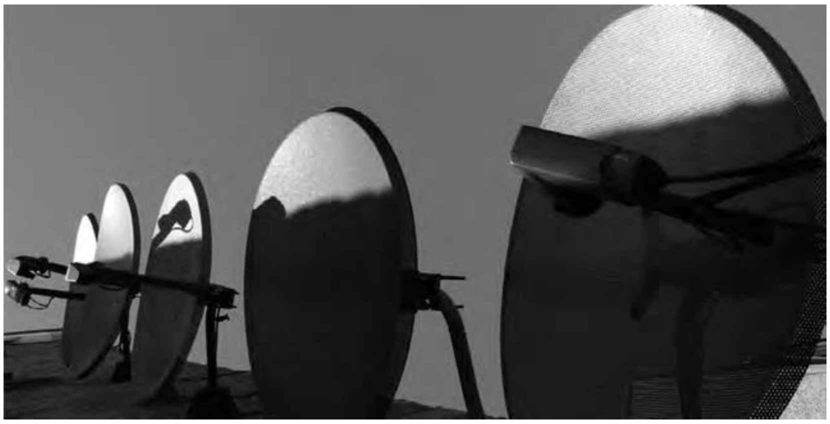


Figure 1 An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

Curved antennas, such as the ones shown in **Figure 1** are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in **Figure 2**.

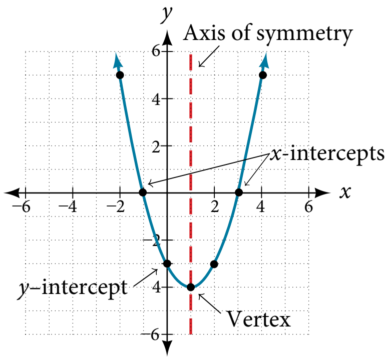


Figure 2

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 1 Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in **Figure 3**.

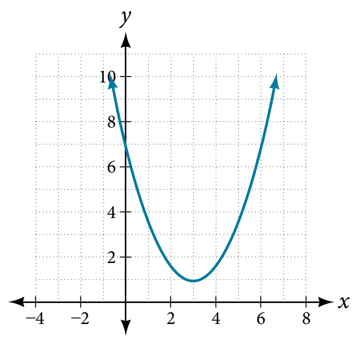


Figure 3

Solution The vertex is the turning point of the graph. We can see that the vertex is at (3, 1). Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at (0, 7) so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\dfrac{b}{2a}$. If we use the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\dfrac{b}{2a}$, the equation for the axis of symmetry.

Figure 4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\dfrac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

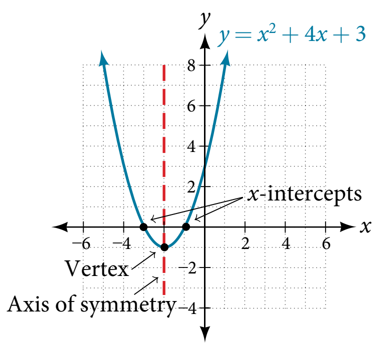


Figure 4

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. **Figure 5** represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

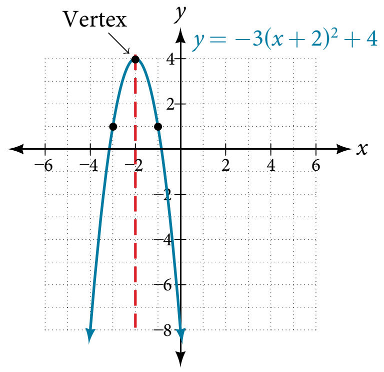


Figure 5

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. **Figure 6** is the graph of this basic function.

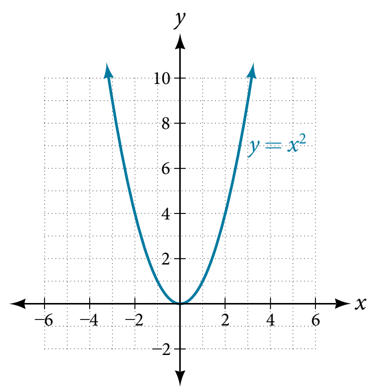


Figure 6

If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In **Figure 5**, $k > 0$, so the graph is shifted 4 units upward. If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In **Figure 5**, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In **Figure 5**, $|a| > 1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

$$a(x - h)^2 + k = ax^2 + bx + c$$

$$ax^2 - 2ahx + (ah^2 + k) = ax^2 + bx + c$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah = b, \text{ so } h = -\frac{b}{2a}.$$

This is the axis of symmetry we defined earlier. Setting the constant terms equal:

$$ah^2 + k = c$$
$$k = c - ah^2$$
$$= c - a\left(\frac{b}{2a}\right)^2$$
$$= c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h) = k$.

forms of quadratic functions

A quadratic function is a polynomial function of degree two. The graph of a quadratic function is a parabola.

The **general form of a quadratic function** is $f(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$.

The **standard form of a quadratic function** is $f(x) = a(x - h)^2 + k$ where $a \neq 0$.

The vertex (h, k) is located at

$$h = -\frac{b}{2a},\ k = f(h) = f\left(\frac{-b}{2a}\right).$$

How To...

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x) = a(x - h)^2 + k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a > 0$. If the parabola opens down, $a < 0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

Example 2 Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in **Figure 7** as a transformation of $f(x) = x^2$, and then expand the formula, and simplify terms to write the equation in general form.

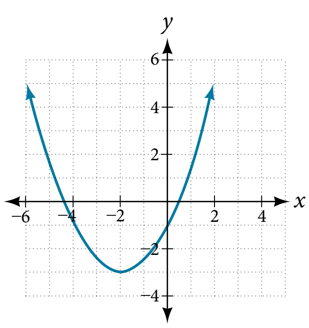


Figure 7

Solution We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x - (-2))^2 - 3 = a(x + 2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0 + 2)^2 - 3$$
$$2 = 4a$$
$$a = \frac{1}{2}$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x + 2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} g(x) &= \frac{1}{2}(x + 2)^2 - 3 \\ &= \frac{1}{2}(x + 2)(x + 2) - 3 \\ &= \frac{1}{2}(x^2 + 4x + 4) - 3 \\ &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis *We can check our work using the table feature on a graphing utility. First enter* $\mathbf{Y1 = \frac{1}{2}(x + 2)^2 - 3}$. *Next, select* ***TBLSET****, then use* $\mathbf{TblStart = -6}$ *and* $\mathbf{\Delta Tbl = 2}$*, and select* ***TABLE****. See* ***Table 1****.*

x	-6	-4	-2	0	2
y	5	-1	-3	-1	5

Table 1

The ordered pairs in the table correspond to points on the graph.

Try It #1

A coordinate grid has been superimposed over the quadratic path of a basketball in **Figure 8** Find an equation for the path of the ball. Does the shooter make the basket?

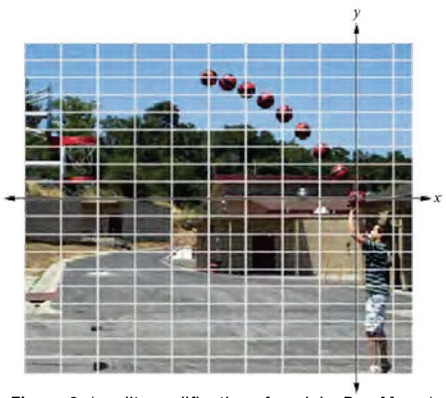


Figure 8 (credit: modification of work by Dan Meyer)

How To…

Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example 3 **Finding the Vertex of a Quadratic Function**

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution The horizontal coordinate of the vertex will be at

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(2)} \\ &= \frac{6}{4} \\ &= \frac{3}{2} \end{aligned}$$

The vertical coordinate of the vertex will be at

$$\begin{aligned} k &= f(h) \\ &= f\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \\ &= \frac{5}{2} \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic. First, find the horizontal coordinate of the vertex. Then find the vertical coordinate of the vertex. Substitute the values into standard form, using the "a" from the general form.

$$f(x) = ax^2 + bx + c$$
$$f(x) = 2x^2 - 6x + 7$$

The standard form of a quadratic function prior to writing the function then becomes the following:

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Analysis *One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, k, and where it occurs, x.*

Try It #2

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

domain and range of a quadratic function

The domain of any quadratic function is all real numbers unless the context of the function presents some restrictions. The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $\left[f\left(-\frac{b}{2a}\right), \infty\right)$.

The range of a quadratic function written in general form with a negative a value is $f(x) \leq f\left(-\frac{b}{2a}\right)$, or $\left(-\infty, f\left(-\frac{b}{2a}\right)\right]$.

The range of a quadratic function written in standard form $f(x) = a(x - h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To…

Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example 4 Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{9}{2(-5)} \\ &= \frac{9}{10} \end{aligned}$$

The maximum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{9}{10}\right) &= -5\left(\frac{9}{10}\right)^2 + 9\left(\frac{9}{10}\right) - 1 \\ &= \frac{61}{20} \end{aligned}$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Try It #3

Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in **Figure 9**.

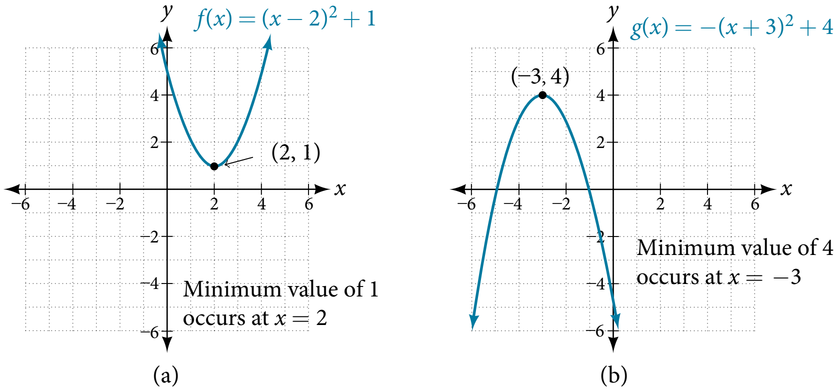


Figure 9

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 5 Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.

b. What dimensions should she make her garden to maximize the enclosed area?

Solution Let's use a diagram such as **Figure 10** to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

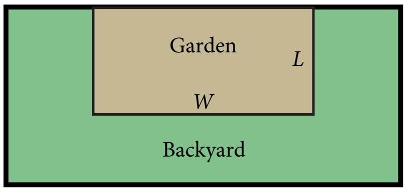


Figure 10

a. We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, W, in terms of L.

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L)$$
$$A(L) = 80L - 2L^2$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L) = -2L^2 + 80L.$$

b. The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$h = -\frac{b}{2a} \qquad\qquad k = A(20)$$
$$h = -\frac{80}{2(-2)} \qquad\qquad = 80(20) - 2(20)^2$$
$$= 20 \qquad \text{and} \qquad = 800$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis *This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in* ***Figure 11****.*

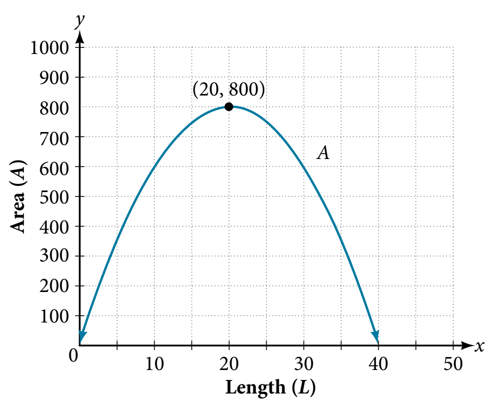


Figure 11

How To...

Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for a revenue function.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example 6 Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of \$30. Market research has suggested that if the owners raise the price to \$32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation Revenue $= pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84{,}000$. We also know that if the price rises to \$32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79{,}000$. From this we can find a linear equation relating the two quantities. The slope will be

$$\begin{aligned} m &= \frac{79{,}000 - 84{,}000}{32 - 30} \\ &= \frac{-5{,}000}{2} \\ &= -2{,}500 \end{aligned}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$\begin{aligned} Q &= -2{,}500p + b && \text{Substitute in the point } Q = 84{,}000 \text{ and } p = 30 \\ 84{,}000 &= -2{,}500(30) + b && \text{Solve for } b \\ b &= 159{,}000 \end{aligned}$$

This gives us the linear equation $Q = -2{,}500p + 159{,}000$ relating cost and subscribers. We now return to our revenue equation.

$$\begin{aligned} \text{Revenue} &= pQ \\ \text{Revenue} &= p(-2{,}500p + 159{,}000) \\ \text{Revenue} &= -2{,}500p^2 + 159{,}000p \end{aligned}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$\begin{aligned} h &= -\frac{159{,}000}{2(-2{,}500)} \\ &= 31.8 \end{aligned}$$

The model tells us that the maximum revenue will occur if the newspaper charges \$31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\begin{aligned} \text{maximum revenue} &= -2{,}500(31.8)^2 + 159{,}000(31.8) \\ &= 2{,}528{,}100 \end{aligned}$$

Analysis *This could also be solved by graphing the quadratic as in* ***Figure 12****. We can see the maximum revenue on a graph of the quadratic function.*

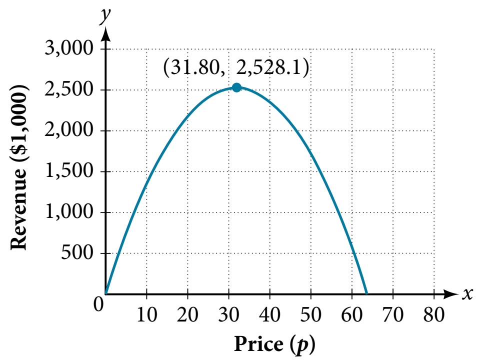


Figure 12

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in **Figure 13** that the number of x-intercepts can vary depending upon the location of the graph.

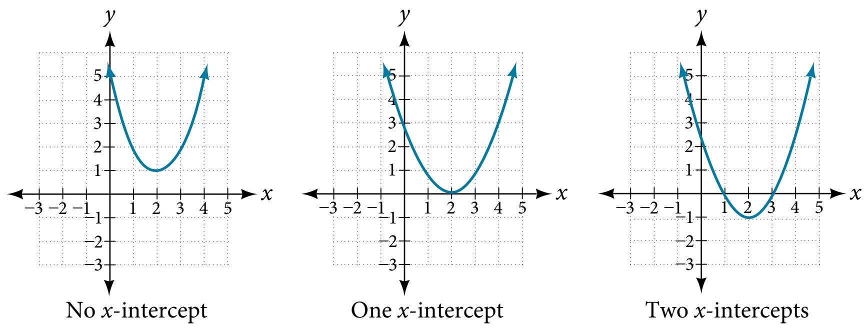


Figure 13 Number of x-intercepts of a parabola

How To...

Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the x-intercepts.

Example 7 Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution We find the y-intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 3(0)^2 + 5(0) - 2 \\ &= -2 \end{aligned}$$

So the y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2)$$

$$0 = 3x - 1 \qquad 0 = x + 2$$

$$x = \frac{1}{3} \qquad \text{or} \qquad x = -2$$

So the x-intercepts are at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$.

Analysis *By graphing the function, we can confirm that the graph crosses the y-axis at* $(0, -2)$. *We can also confirm that the graph crosses the x-axis at* $\left(\frac{1}{3}, 0\right)$ *and* $(-2, 0)$. *See* ***Figure 14.***

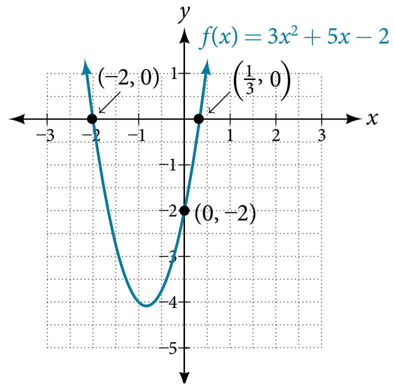


Figure 14

Rewriting Quadratics in Standard Form

In **Example 7**, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

How To...

Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x-intercepts.

Example 8 Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x - h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\frac{b}{2a} & \qquad k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x + 1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x + 1)^2 - 6 \\ 6 &= 2(x + 1)^2 \\ 3 &= (x + 1)^2 \\ x + 1 &= \pm\sqrt{3} \\ x &= -1 \pm\sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$.

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See **Figure 15**.

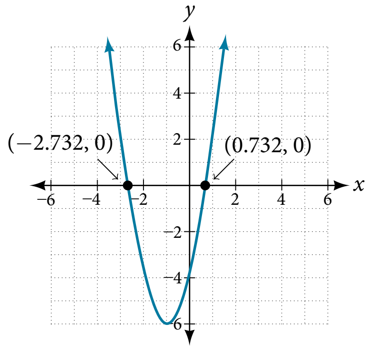


Figure 15

Analysis *We could have achieved the same results using the quadratic formula. Identify* $a = 2$, $b = 4$, *and* $c = -4$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-4)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{48}}{4}$$

$$x = \frac{-4 \pm \sqrt{3(16)}}{4}$$

$$x = -1 \pm \sqrt{3}$$

So the x-intercepts occur at $(-1 - \sqrt{3}, 0)$ *and* $(-1 + \sqrt{3}, 0)$.

Try It #4

In a separate **Try It**, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Example 9 Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

a. When does the ball reach the maximum height?
b. What is the maximum height of the ball?
c. When does the ball hit the ground?

Solution

a. The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)}$$

$$= \frac{80}{32}$$

$$= \frac{5}{2}$$

$$= 2.5$$

The ball reaches a maximum height after 2.5 seconds.

b. To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$\begin{aligned} k &= H\left(-\frac{b}{2a}\right) \\ &= H(2.5) \\ &= -16(2.5)^2 + 80(2.5) + 40 \\ &= 140 \end{aligned}$$

The ball reaches a maximum height of 140 feet.

c. To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$. We use the quadratic formula.

$$\begin{aligned} t &= \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \\ &= \frac{-80 \pm \sqrt{8960}}{-32} \end{aligned}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See **Figure 16.**

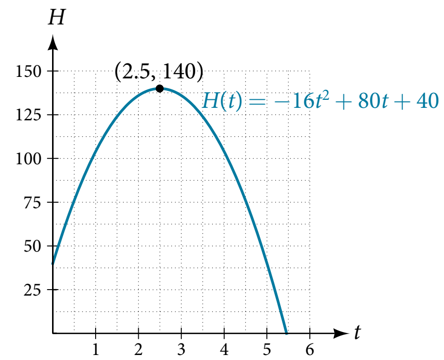


Figure 16

Notice that the graph does not represent the physical path of the ball upward and downward. Keep quantities on each axis in mind while interpreting the graph.

Try It #5

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation
$H(t) = -16t^2 + 96t + 112.$

a. When does the rock reach the maximum height?

b. What is the maximum height of the rock?

c. When does the rock hit the ocean?

Access these online resources for additional instruction and practice with quadratic equations.

- Graphing Quadratic Functions in General Form (http://openstaxcollege.org/l/graphquadgen)
- Graphing Quadratic Functions in Standard Form (http://openstaxcollege.org/l/graphquadstan)
- Quadratic Function Review (http://openstaxcollege.org/l/quadfuncrev)
- Characteristics of a Quadratic Function (http://openstaxcollege.org/l/characterquad)

5.1 SECTION EXERCISES

VERBAL

1. Explain the advantage of writing a quadratic function in standard form.

2. How can the vertex of a parabola be used in solving real-world problems?

3. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

4. What is another name for the standard form of a quadratic function?

5. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

ALGEBRAIC

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

6. $f(x) = x^2 - 12x + 32$

7. $g(x) = x^2 + 2x - 3$

8. $f(x) = x^2 - x$

9. $f(x) = x^2 + 5x - 2$

10. $h(x) = 2x^2 + 8x - 10$

11. $k(x) = 3x^2 - 6x - 9$

12. $f(x) = 2x^2 - 6x$

13. $f(x) = 3x^2 - 5x - 1$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

14. $y(x) = 2x^2 + 10x + 12$

15. $f(x) = 2x^2 - 10x + 4$

16. $f(x) = -x^2 + 4x + 3$

17. $f(x) = 4x^2 + x - 1$

18. $h(t) = -4t^2 + 6t - 1$

19. $f(x) = \frac{1}{2}x^2 + 3x + 1$

20. $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

21. $f(x) = (x - 3)^2 + 2$

22. $f(x) = -2(x + 3)^2 - 6$

23. $f(x) = x^2 + 6x + 4$

24. $f(x) = 2x^2 - 4x + 2$

25. $k(x) = 3x^2 - 6x - 9$

For the following exercises, use the vertex (h, k) and a point on the graph (x, y) to find the general form of the equation of the quadratic function.

26. $(h, k) = (2, 0), (x, y) = (4, 4)$

27. $(h, k) = (-2, -1), (x, y) = (-4, 3)$

28. $(h, k) = (0, 1), (x, y) = (2, 5)$

29. $(h, k) = (2, 3), (x, y) = (5, 12)$

30. $(h, k) = (-5, 3), (x, y) = (2, 9)$

31. $(h, k) = (3, 2), (x, y) = (10, 1)$

32. $(h, k) = (0, 1), (x, y) = (1, 0)$

33. $(h, k) = (1, 0), (x, y) = (0, 1)$

GRAPHICAL

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

34. $f(x) = x^2 - 2x$

35. $f(x) = x^2 - 6x - 1$

36. $f(x) = x^2 - 5x - 6$

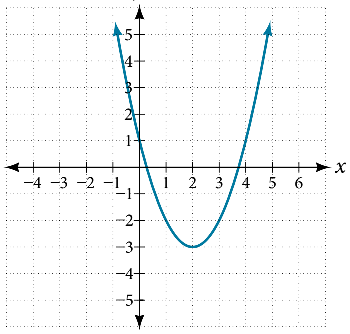
37. $f(x) = x^2 - 7x + 3$

38. $f(x) = -2x^2 + 5x - 8$

39. $f(x) = 4x^2 - 12x - 3$

For the following exercises, write the equation for the graphed function.

40.



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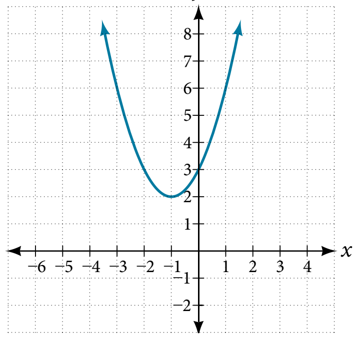

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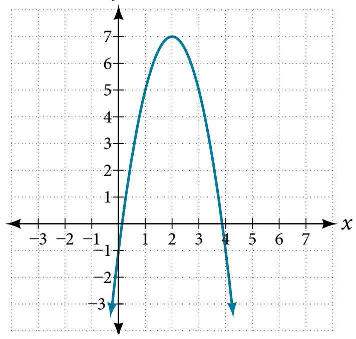

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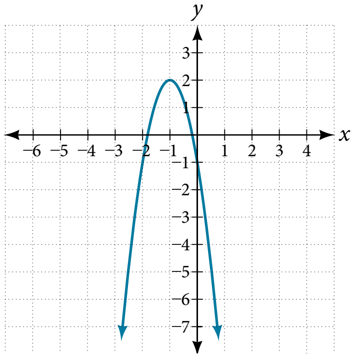

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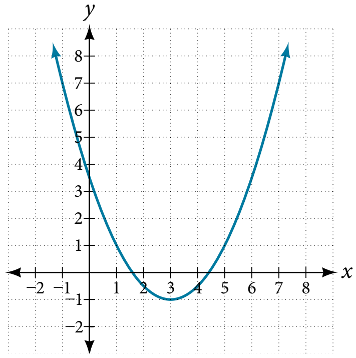

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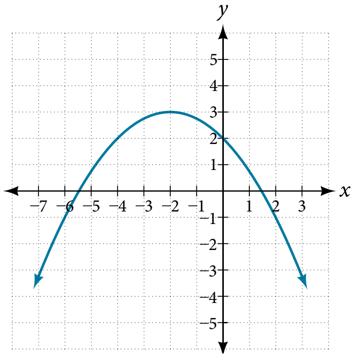


NUMERIC

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

46.

x	−2	−1	0	1	2
y	5	2	1	2	5

47.

x	−2	−1	0	1	2
y	1	0	1	4	9

48.

x	−2	−1	0	1	2
y	−2	1	2	1	−2

49.

x	−2	−1	0	1	2
y	−8	−3	0	1	0

50.

x	−2	−1	0	1	2
y	8	2	0	2	8

TECHNOLOGY

For the following exercises, use a calculator to find the answer.

51. Graph on the same set of axes the functions $f(x) = x^2$, $f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?

52. Graph on the same set of axes $f(x) = x^2$, $f(x) = x^2 + 2$ and $f(x) = x^2$, $f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

53. Graph on the same set of axes $f(x) = x^2$, $f(x) = (x - 2)^2$, $f(x - 3)^2$, and $f(x) = (x + 4)^2$. What appears to be the effect of adding or subtracting those numbers?

54. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = \frac{-32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use the [**TRACE**] feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

55. A suspension bridge can be modeled by the quadratic function $h(x) = 0.0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the [**TRACE**] feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

EXTENSIONS

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

56. Vertex $(1, -2)$, opens up.

57. Vertex $(-1, 2)$ opens down.

58. Vertex $(-5, 11)$, opens down.

59. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

60. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the y-axis.

61. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y-axis.

62. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y-axis.

63. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the y-axis.

64. Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y-axis.

65. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

REAL-WORLD APPLICATIONS

66. Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.

67. Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

68. Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.

69. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

70. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

71. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

72. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

73. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

74. A soccer stadium holds 62,000 spectators. With a ticket price of \$11, the average attendance has been 26,000. When the price dropped to \$9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

75. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

LEARNING OBJECTIVES

In this section, you will:

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

5.4 DIVIDING POLYNOMIALS

Figure 1 Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m.[15] We can easily find the volume using elementary geometry.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73{,}800 \end{aligned}$$

So the volume is 73,800 cubic meters (m^3). Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73{,}800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$. To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15}\downarrow \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$

Step 2: Bring down the 8

Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\ R\ 1$ or $59\frac{1}{3}$

15. National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historyculture/lincoln-memorial-building-statistics.htm. Accessed 4/3/2014/

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor} \cdot \text{quotient}) + \text{remainder} \\ 178 &= (3 \cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$x+2\overline{)2x^3-3x^2+\ \ 4x+5}$$

Set up the division problem.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+\ \ 4x+5} \end{array}$$

$2x^3$ divided by x is $2x^2$.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+\ \ 4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+\ \ 4x \end{array}$$

Multiply $x + 2$ by $2x^2$.
Subtract.
Bring down the next term.

$$\begin{array}{r} 2x^2-\ \ 7x \\ x+2\overline{)2x^3-3x^2+\ \ 4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+\ \ 4x \\ \underline{-(-7x^2+14x)} \\ 18x+5 \end{array}$$

$-7x^2$ divided by x is $-7x$.
Multiply $x + 2$ by $-7x$.
Subtract. Bring down the next term.

$$\begin{array}{r} 2x^2-\ \ 7x+18 \\ x+2\overline{)2x^3-3x^2+\ \ 4x+5} \\ \underline{-(2x^3+4x^2)} \\ -7x^2+\ \ 4x \\ \underline{-(-7x^2+14x)} \\ 18x+5 \\ \underline{-18x+36} \\ -31 \end{array}$$

$18x$ divided by x is 18.
Multiply $x + 2$ by 18.
Subtract.

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2} = 2x^2 - 7x + 18 - \frac{31}{x+2}$$

or

$$\frac{2x^3-3x^2+4x+5}{x+2} = (x+2)(2x^2 - 7x + 18) - 31$$

We can identify the dividend, the divisor, the quotient, and the remainder.

$$\underbrace{2x^3-3x^2+4x+5}_{\text{Dividend}} = \underbrace{(x+2)}_{\text{Divisor}}\underbrace{(2x^2-7x+18)}_{\text{Quotient}} + \underbrace{(-31)}_{\text{Remainder}}$$

Writing the result in this manner illustrates the Division Algorithm.

> ***the Division Algorithm***
>
> The **Division Algorithm** states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that
>
> $$f(x) = d(x)q(x) + r(x)$$
>
> $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.
>
> If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

How To…

Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example 1 Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution

$$x + 1\overline{)5x^2 + 3x - 2}$$
Set up division problem.

$$\begin{array}{r} 5x \phantom{{}-2} \\ x + 1\overline{)5x^2 + 3x - 2} \end{array}$$
$5x^2$ divided by x is $5x$.

$$\begin{array}{r} 5x \phantom{{}-2} \\ x + 1\overline{)5x^2 + 3x - 2} \\ \underline{-(5x^2 + 5x)} \phantom{{}-2} \\ -2x - 2 \end{array}$$
Multiply $x + 1$ by $5x$.
Subtract. Bring down the next term.

$$\begin{array}{r} 5x - 2 \\ x + 1\overline{)5x^2 + 3x - 2} \\ \underline{-(5x^2 + 5x)} \phantom{{}-2} \\ -2x - 2 \\ \underline{-(-2x - 2)} \\ 0 \end{array}$$
$-2x$ divided by x is -2.
Multiply $x + 1$ by -2.
Subtract.

The quotient is $5x - 2$. The remainder is 0. We write the result as

$$\frac{5x^2 + 3x - 2}{x + 1} = 5x - 2$$

or

$$5x^2 + 3x - 2 = (x + 1)(5x - 2)$$

Analysis *This division problem had a remainder of* 0. *This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.*

Example 2 Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$.

Solution

$$\begin{array}{rrl} & 2x^2 + \quad 5x - 7 & \quad 6x^3 \text{ divided by } 3x \text{ is } 2x^2. \\ 3x - 2\overline{)6x^3 + 11x^2 - 31x + 15} & & \\ \underline{-(6x^3 - \quad 4x^2)} & & \quad \text{Multiply } 3x - 2 \text{ by } 2x^2. \\ 15x^2 - 31x & & \quad \text{Subtract. Bring down the next term. } 15x^2 \text{ divided by } 3x \text{ is } 5x. \\ \underline{-(15x^2 - 10x)} & & \quad \text{Multiply } 3x - 2 \text{ by } 5x. \\ -21x + 15 & & \quad \text{Subtract. Bring down the next term. } -21x \text{ divided by } 3x \text{ is } -7. \\ \underline{-(-21x + 14)} & & \quad \text{Multiply } 3x - 2 \text{ by } -7. \\ 1 & & \quad \text{Subtract. The remainder is 1.} \end{array}$$

There is a remainder of 1. We can express the result as:

$$\frac{6x^3 + 11x^2 - 31x + 15}{3x - 2} = 2x^2 + 5x - 7 + \frac{1}{3x - 2}$$

Analysis *We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.*

$$(3x - 2)(2x^2 + 5x - 7) + 1 = 6x^3 + 11x^2 - 31x + 15$$

Notice, as we write our result,

- *the dividend is* $6x^3 + 11x^2 - 31x + 15$
- *the divisor is* $3x - 2$
- *the quotient is* $2x^2 + 5x - 7$
- *the remainder is* 1

Try It #1

Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm.

The final form of the process looked like this:

$$\begin{array}{r} 2x^2 + \quad x + 18 \\ x + 2\overline{)2x^3 - 3x^2 + \ 4x + 5} \\ \underline{-(2x^3 + 4x^2)} \qquad\qquad \\ -7x^2 + \ 4x \qquad \\ \underline{-(-7x^2 - 14x)} \qquad \\ 18x + 5 \\ \underline{-(18x + 36)} \\ -31 \end{array}$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{rrrrr} 2) & 2 & -3 & 4 & 5 \\ & -2 & -4 & & \\ \hline & & -7 & 14 & \\ \hline & & & 18 & -36 \\ \hline & & & & -31 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to −2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 - 7x + 18$ and the remainder is −31. The process will be made more clear in **Example 3**.

synthetic division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$ where k is a real number. In **synthetic division**, only the coefficients are used in the division process.

How To...

Given two polynomials, use synthetic division to divide.

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example 3 Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ \hline \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k. Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis *Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.*

$$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36$$

Example 4 Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by -2, and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

The result is $4x^2 + 2x - 10$. The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

Analysis *The graph of the polynomial function* $f(x) = 4x^3 + 10x^2 - 6x - 20$ *in* ***Figure 2*** *shows a zero at* $x = k = -2$. *This confirms that* $x + 2$ *is a factor of* $4x^3 + 10x^2 - 6x - 20$.

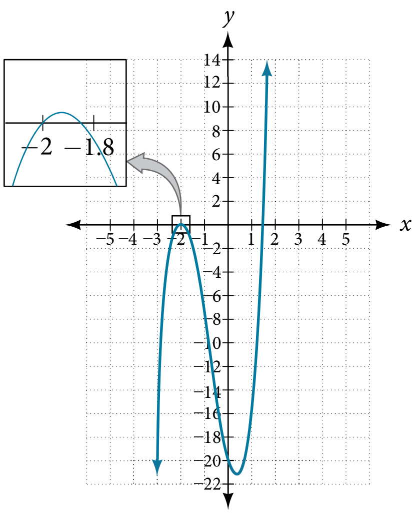


Figure 2

Example 5 Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3 + x^2 + 8x + 8 + \dfrac{2}{x - 1}$.

Try It #2

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Example 6 Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height, t, of the solid.

Solution There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in **Figure 3**.

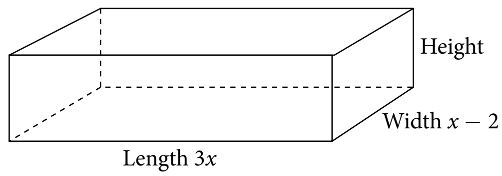


Figure 3

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$V = l \cdot w \cdot h$$
$$3x^4 - 3x^3 - 33x^2 + 54x = 3x \cdot (x - 2) \cdot h$$

To solve for h, first divide both sides by $3x$.

$$\frac{3x \cdot (x-2) \cdot h}{3x} = \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x}$$
$$(x - 2)h = x^3 - x^2 - 11x + 18$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2}$$

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

Try It #3

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

Access these online resources for additional instruction and practice with polynomial division.

- Dividing a Trinomial by a Binomial Using Long Division (http://openstaxcollege.org/l/dividetribild)
- Dividing a Polynomial by a Binomial Using Long Division (http://openstaxcollege.org/l/dividepolybild)
- Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division (http://openstaxcollege.org/l/dividepolybisd2)
- Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division (http://openstaxcollege.org/l/dividepolybisd4)

5.4 SECTION EXERCISES

VERBAL

1. If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

2. If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

ALGEBRAIC

For the following exercises, use long division to divide. Specify the quotient and the remainder.

3. $(x^2 + 5x - 1) \div (x - 1)$

4. $(2x^2 - 9x - 5) \div (x - 5)$

5. $(3x^2 + 23x + 14) \div (x + 7)$

6. $(4x^2 - 10x + 6) \div (4x + 2)$

7. $(6x^2 - 25x - 25) \div (6x + 5)$

8. $(-x^2 - 1) \div (x + 1)$

9. $(2x^2 - 3x + 2) \div (x + 2)$

10. $(x^3 - 126) \div (x - 5)$

11. $(3x^2 - 5x + 4) \div (3x + 1)$

12. $(x^3 - 3x^2 + 5x - 6) \div (x - 2)$

13. $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

For the following exercises, use synthetic division to find the quotient.

14. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

15. $(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$

16. $(6x^3 - 10x^2 - 7x - 15) \div (x + 1)$

17. $(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$

18. $(9x^3 - 9x^2 + 18x + 5) \div (3x - 1)$

19. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

20. $(-6x^3 + x^2 - 4) \div (2x - 3)$

21. $(2x^3 + 7x^2 - 13x - 3) \div (2x - 3)$

22. $(3x^3 - 5x^2 + 2x + 3) \div (x + 2)$

23. $(4x^3 - 5x^2 + 13) \div (x + 4)$

24. $(x^3 - 3x + 2) \div (x + 2)$

25. $(x^3 - 21x^2 + 147x - 343) \div (x - 7)$

26. $(x^3 - 15x^2 + 75x - 125) \div (x - 5)$

27. $(9x^3 - x + 2) \div (3x - 1)$

28. $(6x^3 - x^2 + 5x + 2) \div (3x + 1)$

29. $(x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1)$

30. $(x^4 - 3x^2 + 1) \div (x - 1)$

31. $(x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3)$

32. $(x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2)$

33. $(x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2)$

34. $(x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5)$

35. $(x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3)$

36. $(4x^4 - 2x^3 - 4x + 2) \div (2x - 1)$

37. $(4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1)$

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

38. $x - 2,\ 4x^3 - 3x^2 - 8x + 4$

39. $x - 2,\ 3x^4 - 6x^3 - 5x + 10$

40. $x + 3,\ -4x^3 + 5x^2 + 8$

41. $x - 2,\ 4x^4 - 15x^2 - 4$

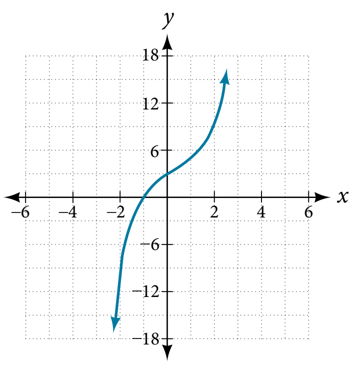
42. $x - \frac{1}{2},\ 2x^4 - x^3 + 2x - 1$

43. $x + \frac{1}{3},\ 3x^4 + x^3 - 3x + 1$

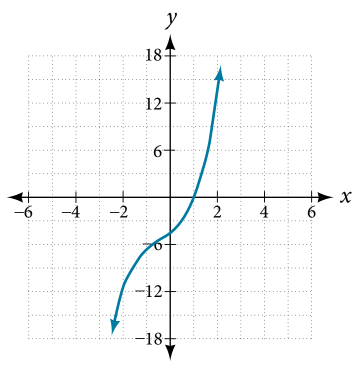
GRAPHICAL

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

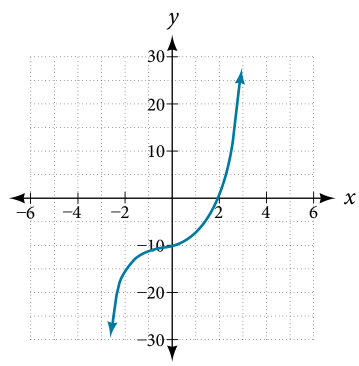
44. Factor is $x^2 - x + 3$



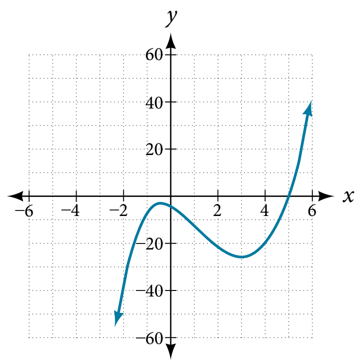
45. Factor is $x^2 + 2x + 4$



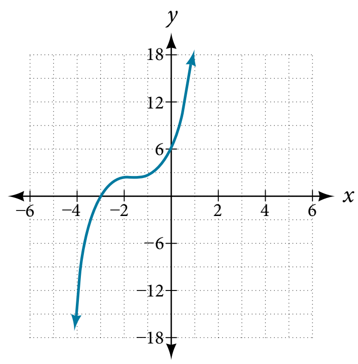
46. Factor is $x^2 + 2x + 5$



47. Factor is $x^2 + x + 1$



48. Factor is $x^2 + 2x + 2$



For the following exercises, use synthetic division to find the quotient and remainder.

49. $\dfrac{4x^3 - 33}{x - 2}$

50. $\dfrac{2x^3 + 25}{x + 3}$

51. $\dfrac{3x^3 + 2x - 5}{x - 1}$

52. $\dfrac{-4x^3 - x^2 - 12}{x + 4}$

53. $\dfrac{x^4 - 22}{x + 2}$

TECHNOLOGY

For the following exercises, use a calculator with CAS to answer the questions.

54. Consider $\dfrac{x^k - 1}{x - 1}$ with $k = 1, 2, 3$. What do you expect the result to be if $k = 4$?

55. Consider $\dfrac{x^k + 1}{x + 1}$ for $k = 1, 3, 5$. What do you expect the result to be if $k = 7$?

56. Consider $\dfrac{x^4 - k^4}{x - k}$ for $k = 1, 2, 3$. What do you expect the result to be if $k = 4$?

57. Consider $\dfrac{x^k}{x + 1}$ with $k = 1, 2, 3$. What do you expect the result to be if $k = 4$?

58. Consider $\dfrac{x^k}{x - 1}$ with $k = 1, 2, 3$. What do you expect the result to be if $k = 4$?

EXTENSIONS

For the following exercises, use synthetic division to determine the quotient involving a complex number.

59. $\dfrac{x + 1}{x - i}$

60. $\dfrac{x^2 + 1}{x - i}$

61. $\dfrac{x + 1}{x + i}$

62. $\dfrac{x^2 + 1}{x + i}$

63. $\dfrac{x^3 + 1}{x - i}$

REAL-WORLD APPLICATIONS

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

64. Length is $x + 5$, area is $2x^2 + 9x - 5$.

65. Length is $2x + 5$, area is $4x^3 + 10x^2 + 6x + 15$

66. Length is $3x - 4$, area is $6x^4 - 8x^3 + 9x^2 - 9x - 4$

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

67. Volume is $12x^3 + 20x^2 - 21x - 36$, length is $2x + 3$, width is $3x - 4$.

68. Volume is $18x^3 - 21x^2 - 40x + 48$, length is $3x - 4$, width is $3x - 4$.

69. Volume is $10x^3 + 27x^2 + 2x - 24$, length is $5x - 4$, width is $2x + 3$.

70. Volume is $10x^3 + 30x^2 - 8x - 24$, length is 2, width is $x + 3$.

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

71. Volume is $\pi(25x^3 - 65x^2 - 29x - 3)$, radius is $5x + 1$.

72. Volume is $\pi(4x^3 + 12x^2 - 15x - 50)$, radius is $2x + 5$.

73. Volume is $\pi(3x^4 + 24x^3 + 46x^2 - 16x - 32)$, radius is $x + 4$.

LEARNING OBJECTIVES

In this section, you will:

- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

5.8 MODELING USING VARIATION

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for \$4,600, she will earn \$736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

Solving Direct Variation Problems

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, e, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See **Table 1**.

s, sales price	$e = 0.16s$	Interpretation
\$4,600	$e = 0.16(4{,}600) = 736$	A sale of *a* \$4,600 vehicle results in \$736 earnings.
\$9,200	$e = 0.16(9{,}200) = 1{,}472$	A sale of *a* \$9,200 vehicle results in \$1472 earnings.
\$18,400	$e = 0.16(18{,}400) = 2{,}944$	A sale of *a* \$18,400 vehicle results in \$2944 earnings.

Table 1

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from \$4,600 to \$9,200, and we double the earnings from \$736 to \$1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value k is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$. We saw functions like this one when we discussed power functions.

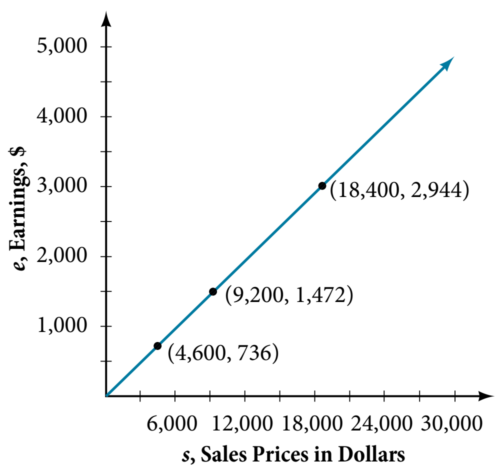


Figure 1

direct variation

If x and y are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is **direct variation** and y **varies directly** with, or is proportional to, the nth power of x. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the **constant of variation**, which help defines the relationship between the variables.

How To...

Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 1 Solving a Direct Variation Problem

The quantity y varies directly with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution The general formula for direct variation with a cube is $y = kx^3$. The constant can be found by dividing y by the cube of x.

$$k = \frac{y}{x^3} = \frac{25}{2^3} = \frac{25}{8}$$

Now use the constant to write an equation that represents this relationship.

$$y = \frac{25}{8}x^3$$

Substitute $x = 6$ and solve for y.

$$y = \frac{25}{8}(6)^3 = 675$$

Analysis *The graph of this equation is a simple cubic, as shown in **Figure 2**.*

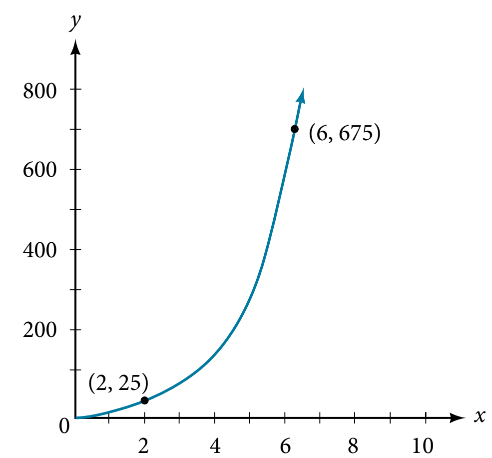


Figure 2

Q & A…

Do the graphs of all direct variation equations look like Example 1?

No. Direct variation equations are power functions—they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through (0,0).

Try It #1

The quantity y varies directly with the square of x. If $y = 24$ when $x = 3$, find y when x is 4.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. Between the depths of 250 feet and 500 feet, the formula $T = \frac{14{,}000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F. If we create **Table 2**, we observe that, as the depth increases, the water temperature decreases.

d, depth	$T = \frac{14{,}000}{d}$	Interpretation
500 ft	$\frac{14{,}000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.
1,000 ft	$\frac{14{,}000}{1{,}000} = 14$	At a depth of 1,000 ft, the water temperature is 14° F.
2,000 ft	$\frac{14{,}000}{2{,}000} = 7$	At a depth of 2,000 ft, the water temperature is 7° F.

Table 2

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, **Figure 3** depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14{,}000$.

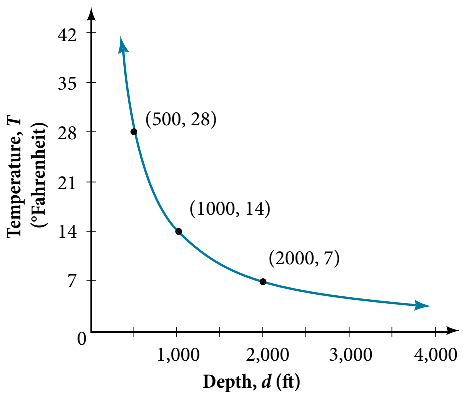


Figure 3

inverse variation

If x and y are related by an equation of the form

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y **varies inversely** with the nth power of x. In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

Example 2 Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

Solution Recall that multiplying speed by time gives distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which the tourist drives, then $vt =$ distance. Because the distance is fixed at 100 miles, $vt = 100$ so $t = \frac{100}{v}$. Because time is a function of velocity, we can write $t(v)$.

$$\begin{aligned} t(v) &= \frac{100}{v} \\ &= 100v^{-1} \end{aligned}$$

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction. We say that the time varies inversely with velocity.

How To...

Given a description of an indirect variation problem, solve for an unknown.

1. Identify the input, x, and the output, y.
2. Determine the constant of variation. You may need to multiply y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Example 3 Solving an Inverse Variation Problem

A quantity y varies inversely with the cube of x. If $y = 25$ when $x = 2$, find y when x is 6.

Solution The general formula for inverse variation with a cube is $y = \frac{k}{x^3}$. The constant can be found by multiplying y by the cube of x.

$$\begin{aligned} k &= x^3 y \\ &= 2^3 \cdot 25 \\ &= 200 \end{aligned}$$

Now we use the constant to write an equation that represents this relationship.

$$y = \frac{k}{x^3},\ k = 200$$

$$y = \frac{200}{x^3}$$

Substitute $x = 6$ and solve for y.

$$\begin{aligned} y &= \frac{200}{6^3} \\ &= \frac{25}{27} \end{aligned}$$

Analysis *The graph of this equation is a rational function, as shown in* ***Figure*** **4**.

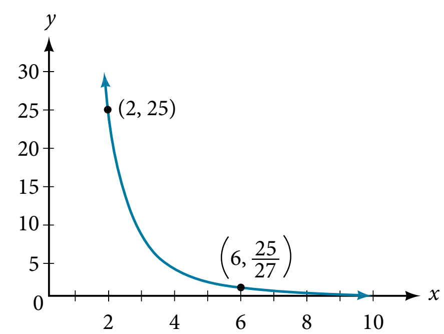


Figure 4

Try It #2

A quantity y varies inversely with the square of x. If $y = 8$ when $x = 3$, find y when x is 4.

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c, cost, varies jointly with the number of students, n, and the distance, d.

> ***joint variation***
>
> Joint variation occurs when a variable varies directly or inversely with multiple variables.
>
> For instance, if x varies directly with both y and z, we have $x = kyz$. If x varies directly with y and inversely with z, we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

Example 4 Solving Problems Involving Joint Variation

A quantity x varies directly with the square of y and inversely with the cube root of z. If $x = 6$ when $y = 2$ and $z = 8$, find x when $y = 1$ and $z = 27$.

Solution Begin by writing an equation to show the relationship between the variables.

$$x = \frac{ky^2}{\sqrt[3]{z}}$$

Substitute $x = 6$, $y = 2$, and $z = 8$ to find the value of the constant k.

$$6 = \frac{k2^2}{\sqrt[3]{8}}$$

$$6 = \frac{4k}{2}$$

$$3 = k$$

Now we can substitute the value of the constant into the equation for the relationship.

$$x = \frac{3y^2}{\sqrt[3]{z}}$$

To find x when $y = 1$ and $z = 27$, we will substitute values for y and z into our equation.

$$x = \frac{3(1)^2}{\sqrt[3]{27}}$$

$$= 1$$

Try It #3

A quantity x varies directly with the square of y and inversely with z. If $x = 40$ when $y = 4$ and $z = 2$, find x when $y = 10$ and $z = 25$.

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation (http://openstaxcollege.org/l/directvariation)
- Inverse Variation (http://openstaxcollege.org/l/inversevariatio)
- Direct and Inverse Variation (http://openstaxcollege.org/l/directinverse)

5.8 SECTION EXERCISES

VERBAL

1. What is true of the appearance of graphs that reflect a direct variation between two variables?

2. If two variables vary inversely, what will an equation representing their relationship look like?

3. Is there a limit to the number of variables that can jointly vary? Explain.

ALGEBRAIC

For the following exercises, write an equation describing the relationship of the given variables.

4. y varies directly as x and when $x = 6$, $y = 12$.

5. y varies directly as the square of x and when $x = 4$, $y = 80$.

6. y varies directly as the square root of x and when $x = 36$, $y = 24$.

7. y varies directly as the cube of x and when $x = 36$, $y = 24$.

8. y varies directly as the cube root of x and when $x = 27$, $y = 15$.

9. y varies directly as the fourth power of x and when $x = 1$, $y = 6$.

10. y varies inversely as x and when $x = 4$, $y = 2$.

11. y varies inversely as the square of x and when $x = 3$, $y = 2$.

12. y varies inversely as the cube of x and when $x = 2$, $y = 5$.

13. y varies inversely as the fourth power of x and when $x = 3$, $y = 1$.

14. y varies inversely as the square root of x and when $x = 25$, $y = 3$.

15. y varies inversely as the cube root of x and when $x = 64$, $y = 5$.

16. y varies jointly with x and z and when $x = 2$ and $z = 3$, $y = 36$.

17. y varies jointly as x, z, and w and when $x = 1$, $z = 2$, $w = 5$, then $y = 100$.

18. y varies jointly as the square of x and the square of z and when $x = 3$ and $z = 4$, then $y = 72$.

19. y varies jointly as x and the square root of z and when $x = 2$ and $z = 25$, then $y = 100$.

20. y varies jointly as the square of x the cube of z and the square root of w. When $x = 1$, $z = 2$, and $w = 36$, then $y = 48$.

21. y varies jointly as x and z and inversely as w. When $x = 3$, $z = 5$, and $w = 6$, then $y = 10$.

22. y varies jointly as the square of x and the square root of z and inversely as the cube of w. When $x = 3$, $z = 4$, and $w = 3$, then $y = 6$.

23. y varies jointly as x and z and inversely as the square root of w and the square of t. When $x = 3$, $z = 1$, $w = 25$, and $t = 2$, then $y = 6$.

NUMERIC

For the following exercises, use the given information to find the unknown value.

24. y varies directly as x. When $x = 3$, then $y = 12$. Find y when $x = 20$.

25. y varies directly as the square of x. When $x = 2$, then $y = 16$. Find y when $x = 8$.

26. y varies directly as the cube of x. When $x = 3$, then $y = 5$. Find y when $x = 4$.

27. y varies directly as the square root of x. When $x = 16$, then $y = 4$. Find y when $x = 36$.

28. y varies directly as the cube root of x. When $x = 125$, then $y = 15$. Find y when $x = 1{,}000$.

29. y varies inversely with x. When $x = 3$, then $y = 2$. Find y when $x = 1$.

30. y varies inversely with the square of x. When $x = 4$, then $y = 3$. Find y when $x = 2$.

31. y varies inversely with the cube of x. When $x = 3$, then $y = 1$. Find y when $x = 1$.

32. y varies inversely with the square root of x. When $x = 64$, then $y = 12$. Find y when $x = 36$.

33. y varies inversely with the cube root of x. When $x = 27$, then $y = 5$. Find y when $x = 125$.

34. y varies jointly as x and z. When $x = 4$ and $z = 2$, then $y = 16$. Find y when $x = 3$ and $z = 3$.

35. y varies jointly as x, z, and w. When $x = 2$, $z = 1$, and $w = 12$, then $y = 72$. Find y when $x = 1$, $z = 2$, and $w = 3$.

36. y varies jointly as x and the square of z. When $x = 2$ and $z = 4$, then $y = 144$. Find y when $x = 4$ and $z = 5$.

37. y varies jointly as the square of x and the square root of z. When $x = 2$ and $z = 9$, then $y = 24$. Find y when $x = 3$ and $z = 25$.

38. y varies jointly as x and z and inversely as w. When $x = 5$, $z = 2$, and $w = 20$, then $y = 4$. Find y when $x = 3$ and $z = 8$, and $w = 48$.

39. y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x = 2$, $z = 2$, and $w = 64$, then $y = 12$. Find y when $x = 1$, $z = 3$, and $w = 4$.

40. y varies jointly as the square of x and of z and inversely as the square root of w and of t. When $x = 2$, $z = 3$, $w = 16$, and $t = 3$, then $y = 1$. Find y when $x = 3$, $z = 2$, $w = 36$, and $t = 5$.

TECHNOLOGY

For the following exercises, use a calculator to graph the equation implied by the given variation.

41. y varies directly with the square of x and when $x = 2$, $y = 3$.

42. y varies directly as the cube of x and when $x = 2$, $y = 4$.

43. y varies directly as the square root of x and when $x = 36$, $y = 2$.

44. y varies inversely with x and when $x = 6$, $y = 2$.

45. y varies inversely as the square of x and when $x = 1$, $y = 4$.

EXTENSIONS

For the following exercises, use Kepler's Law, which states that the square of the time, T, required for a planet to orbit the Sun varies directly with the cube of the mean distance, a, that the planet is from the Sun.

46. Using the Earth's time of 1 year and mean distance of 93 million miles, find the equation relating T and a.

47. Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

48. Using Earth's distance of 150 million kilometers, find the equation relating T and a.

49. Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

50. Using Earth's distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

REAL-WORLD APPLICATIONS

For the following exercises, use the given information to answer the questions.

51. The distance s that an object falls varies directly with the square of the time, t, of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

52. The velocity v of a falling object varies directly to the time, t, of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

53. The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

54. The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

55. The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

56. The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is 0.08 foot-candles at a distance of 3 meters. Find the intensity level at 8 meters.

57. The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

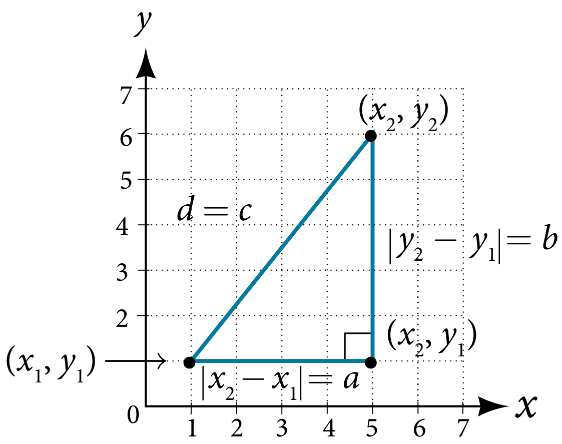
58. The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

59. The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

60. The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 40 kilograms with a velocity of 15 meters per second has a kinetic energy of 1000 joules, find the kinetic energy if the velocity is increased to 20 meters per second.

Using the Distance Formula

Derived from the Pythagorean Theorem, the **distance formula** is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^2 + b^2 = c^2$, is based on a right triangle where a and b are the lengths of the legs adjacent to the right angle, and c is the length of the hypotenuse. See the figure below.



The relationship of sides $|x_2 - x_1|$ and $|y_2 - y_1|$ to side d is the same as that of sides a and b to side c. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3| = 3$.) The symbols $|x_2 - x_1|$ and $|y_2 - y_1|$ indicate that the lengths of the sides of the triangle are positive. To find the length c, take the square root of both sides of the Pythagorean Theorem.

$$c^2 = a^2 + b^2 \to c = \sqrt{a^2 + b^2}$$

It follows that the distance formula is given as

$$d^2 = (x_2 - x_1)^2 + (y^2 - y^1)^2 \to d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We do not have to use the absolute value symbols in this definition because any number squared is non-negative.

The Distance Formula

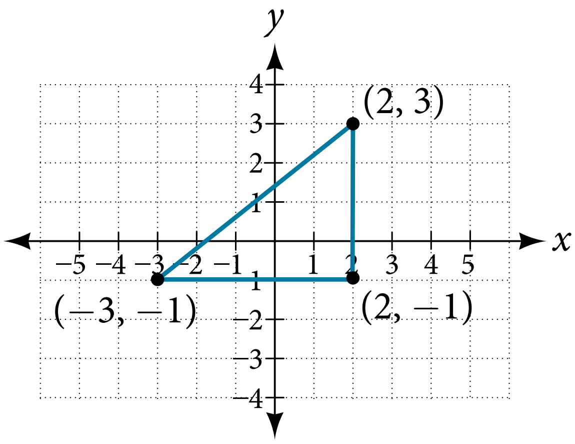
Given endpoints (x_1, y_1) and (x_2, y_2), the distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 14 Finding the Distance between Two Points

Find the distance between the points $(-3, -1)$ and $(2, 3)$.

Solution:
Let us first look at the graph of the two points. Connect the points to form a right triangle as in the figure below.



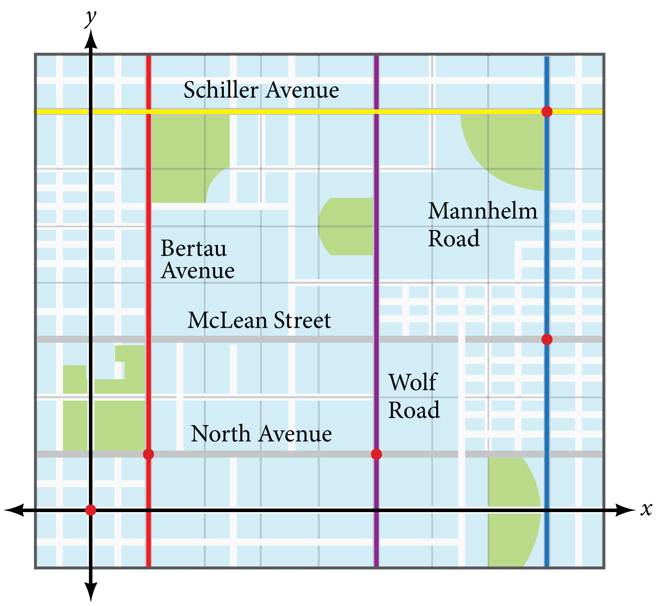
Then, calculate the length of d using the distance formula.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ d &= \sqrt{(2 - (-3))^2 + (3 - (-1))^2} \\ &= \sqrt{(5)^2 + (4)^2} \\ &= \sqrt{25 + 16} \\ &= \sqrt{41} \end{aligned}$$

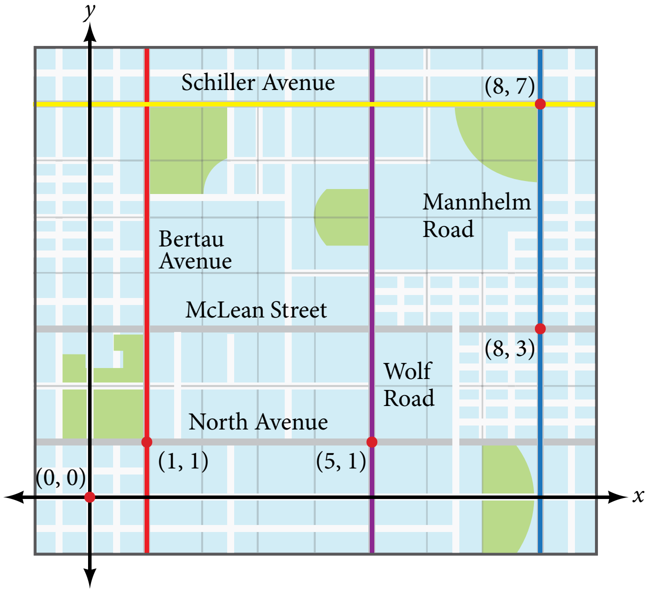
Try It #14: Find the distance between two points: $(1, 4)$ and $(11, 9)$.

Example 15 Finding the Distance Between Two Locations

Tracie set out from Elmhurst, IL, to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in the figure below. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.



Solution: The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1,1)$. The next stop is 5 blocks to the east, so it is at $(5,1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8,3)$. Lastly, she traveled 4 blocks north to $(8,7)$. We can label these points on the grid as in the figure below.



Next, we can calculate the distance. Note that each grid unit represents 1,000 feet.

- From her starting location to her first stop at $(1,1)$, Tracie might have driven north 1,000 feet and then east 1,000 feet, or vice versa. Either way, she drove 2,000 feet to her first stop.
- Her second stop is at $(5,1)$. So from $(1,1)$ to $(5,1)$, Tracie drove east 4,000 feet.
- Her third stop is at $(8,3)$. There are a number of routes from $(5,1)$ to $(8,3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east 3,000 feet and then north 2,000 feet for a total of 5,000 feet.
- Tracies final stop is at $(8,7)$. This is a straight drive north from $(8,3)$ for a total of 4,000 feet. Next, we will add the distances listed in the table below.

From/To	**Number of Feet Driven**
$(0,0)$ to $(1,1)$	2000
$(1,1)$ to $(5,1)$	4000
$(5,1)$ to $(8,3)$	5000
$(8,3)$ to $(8,7)$	4000
Total	15,000

The total distance Tracie drove is 15,000 feet, or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0,0)$ and $(8,7)$.

$$\begin{aligned} d &= \sqrt{(8-0)^2+(7-0)^2} \\ &= \sqrt{64+49} \\ &= \sqrt{113} \\ &\approx 10.63 \text{ units} \end{aligned}$$

At 1,000 feet per grid unit, the distance between Elmhurst, IL, to Franklin Park is 10,630.14 feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8,7)$. Perhaps you have heard the saying "as the crow flies," which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

For each of the following exercises, find the distance between the two points. Simplify your answers, and write the exact answer in simplest radical form for irrational answers.

1. $(4, 1)$ and $(3, 4)$ **2**. $(2, 5)$ and $(7, 4)$ **3**. $(5, 0)$ and $(5, 6)$ **4**. $(4, 3)$ and $(10, 3)$

5. Find the distance between the two points given using your calculator, and round your answer to the nearest hundredth. $(19, 12)$ and $(41, 71)$

6. The coordinates on a map for San Francisco are $(53, 17)$ and those for Sacramento are $(123, 78)$. Note that coordinates represent miles. Find the distance between the cities to the nearest mile.

7. If San Joses coordinates are $(76, -12)$, where the coordinates represent miles, find the distance between San Jose and San Francisco to the nearest mile.

8. A small craft in Lake Ontario sends out a distress signal. The coordinates of the boat in trouble were $(49, 64)$. One rescue boat is at the coordinates $(60, 82)$ and a second Coast Guard craft is at coordinates $(58, 47)$. Assuming both rescue craft travel at the same rate, which one would get to the distressed boat the fastest?

CHAPTER 5 REVIEW

Key Terms

arrow notation a way to represent the local and end behavior of a function by using arrows to indicate that an input or output approaches a value

axis of symmetry a vertical line drawn through the vertex of a parabola, that opens up or down, around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

coefficient a nonzero real number multiplied by a variable raised to an exponent

constant of variation the non-zero value k that helps define the relationship between variables in direct or inverse variation

continuous function a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree the highest power of the variable that occurs in a polynomial

Descartes' Rule of Signs a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

direct variation the relationship between two variables that are a constant multiple of each other; as one quantity increases, so does the other

Division Algorithm given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)\, q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

end behavior the behavior of the graph of a function as the input decreases without bound and increases without bound

Factor Theorem k is a zero of polynomial function $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra a polynomial function with degree greater than 0 has at least one complex zero

general form of a quadratic function the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$.

global maximum highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

global minimum lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

horizontal asymptote a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound.

Intermediate Value Theorem for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

inverse variation the relationship between two variables in which the product of the variables is a constant

inversely proportional a relationship where one quantity is a constant divided by the other quantity; as one quantity increases, the other decreases

invertible function any function that has an inverse function

imaginary number a number in the form bi where $i = \sqrt{-1}$

joint variation a relationship where a variable varies directly or inversely with multiple variables

leading coefficient the coefficient of the leading term

leading term the term containing the highest power of the variable

Linear Factorization Theorem allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number

multiplicity the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x - h)^p$, $x = h$ is a zero of multiplicity p.

polynomial function a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

power function a function that can be represented in the form $f(x) = kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

rational function a function that can be written as the ratio of two polynomials

Rational Zero Theorem the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem if a polynomial $f(x)$ is divided by $x - k$, then the remainder is equal to the value $f(k)$

removable discontinuity a single point at which a function is undefined that, if filled in, would make the function continuous; it appears as a hole on the graph of a function

roots in a given function, the values of x at which $y = 0$, also called zeros

smooth curve a graph with no sharp corners

standard form of a quadratic function the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

synthetic division a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

term of a polynomial function any $a_i x^i$ of a polynomial function in the form $f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$

turning point the location at which the graph of a function changes direction

varies directly a relationship where one quantity is a constant multiplied by the other quantity

varies inversely a relationship where one quantity is a constant divided by the other quantity

vertex the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function another name for the standard form of a quadratic function

vertical asymptote a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a

zeros in a given function, the values of x at which $y = 0$, also called roots

Key Equations

general form of a quadratic function	$f(x) = ax^2 + bx + c$
standard form of a quadratic function	$f(x) = a(x - h)^2 + k$
general form of a polynomial function	$f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$
Division Algorithm	$f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$
Rational Function	$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + ... + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + ... + b_1 x + b_0}, Q(x) \neq 0$
Direct variation	$y = kx^n$, k is a nonzero constant.
Inverse variation	$y = \frac{k}{x^n}$, k is a nonzero constant.

Key Concepts

5.1 Quadratic Functions

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis. See **Example 1**, **Example 7**, and **Example 8**.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph. See **Example 2**.
- The vertex can be found from an equation representing a quadratic function. See **Example 3**.
- The domain of a quadratic function is all real numbers. The range varies with the function. See **Example 4**.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue. See **Example 5** and **Example 6**.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems. See **Example 9.**

5.2 Power Functions and Polynomial Functions

- A power function is a variable base raised to a number power. See **Example 1**.
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd. See **Example 2** and **Example 3**.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power. See **Example 4**.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient. See **Example 5**.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function. See **Example 6** and **Example 7**.
- A polynomial of degree n will have at most n x-intercepts and at most $n - 1$ turning points. See **Example 8**, **Example 9**, **Example 10**, **Example 11**, and **Example 12**.

5.3 Graphs of Polynomial Functions

- Polynomial functions of degree 2 or more are smooth, continuous functions. See **Example 1**.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero. See **Example 2**, **Example 3**, and **Example 4**.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis. See **Example 5**.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts. See **Example 6**.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n - 1$ turning points. See **Example 7**.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n - 1$ turning points. See **Example 8** and **Example 10**.
- Graphing a polynomial function helps to estimate local and global extremas. See **Example 11**.

- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$. See **Example 9**.

5.4 Dividing Polynomials

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree. See **Example 1** and **Example 2**.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form $x - k$. See **Example 3**, **Example 4**, and **Example 5**.
- Polynomial division can be used to solve application problems, including area and volume. See **Example 6**.

5.5 Zeros of Polynomial Functions

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x - k$. this is known as the Remainder Theorem. See **Example 1**.
- According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$. See **Example 2**.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See **Example 3** and **Example 4**.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See **Example 5**.
- According to the Fundamental Theorem, every polynomial function has at least one complex zero. See **Example 6**.
- Every polynomial function with degree greater than 0 has at least one complex zero.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x - c)$, where c is a complex number. See **Example 7**.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer. See **Example 8**.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See **Example 9**.

5.6 Rational Functions

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$. See **Example 1**.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See **Example 2**.
- Application problems involving rates and concentrations often involve rational functions. See **Example 3**.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See **Example 4**.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See **Example 5**.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See **Example 6**.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See **Example 7**, **Example 8**, **Example 9**, and **Example 10**.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See **Example 11**.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form

$$f(x) = a\frac{(x - x_1)^{p_1}(x - x_2)^{p_2}...(x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2}...(x - v_m)^{q_n}}$$

See **Example 12**.

5.7 Inverses and Radical Functions

- The inverse of a quadratic function is a square root function.
- If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. See **Example 1**.
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See **Example 2**.
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one. See **Example 3** and **Example 4**.
- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See **Example 5** and **Example 7**.
- Inverse and radical and functions can be used to solve application problems. See **Example 6** and **Example 8**.

5.8 Modeling Using Variation

- A relationship where one quantity is a constant multiplied by another quantity is called direct variation. See **Example 1**.
- Two variables that are directly proportional to one another will have a constant ratio.
- A relationship where one quantity is a constant divided by another quantity is called inverse variation. See **Example 2**.
- Two variables that are inversely proportional to one another will have a constant multiple. See **Example 3**.
- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation. See **Example 4**.

CHAPTER 5 REVIEW EXERCISES

QUADRATIC FUNCTIONS

For the following exercises, write the quadratic function in standard form. Then, give the vertex and axes intercepts. Finally, graph the function.

1. $f(x) = x^2 - 4x - 5$

2. $f(x) = -2x^2 - 4x$

For the following problems, find the equation of the quadratic function using the given information.

3. The vertex is $(-2, 3)$ and a point on the graph is $(3, 6)$.

4. The vertex is $(-3, 6.5)$ and a point on the graph is $(2, 6)$.

For the following exercises, complete the task.

5. A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is 600 meters, find the dimensions of the plot to have maximum area.

6. An object projected from the ground at a 45 degree angle with initial velocity of 120 feet per second has height, h, in terms of horizontal distance traveled, x, given by $h(x) = \frac{-32}{(120)^2}x^2 + x$. Find the maximum height the object attains.

POWER FUNCTIONS AND POLYNOMIAL FUNCTIONS

For the following exercises, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.

7. $f(x) = 4x^5 - 3x^3 + 2x - 1$

8. $f(x) = 5^{x+1} - x^2$

9. $f(x) = x^2(3 - 6x + x^2)$

For the following exercises, determine end behavior of the polynomial function.

10. $f(x) = 2x^4 + 3x^3 - 5x^2 + 7$

11. $f(x) = 4x^3 - 6x^2 + 2$

12. $f(x) = 2x^2(1 + 3x - x^2)$

GRAPHS OF POLYNOMIAL FUNCTIONS

For the following exercises, find all zeros of the polynomial function, noting multiplicities.

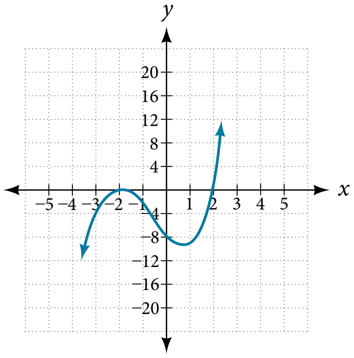
13. $f(x) = (x + 3)^2(2x - 1)(x + 1)^3$

14. $f(x) = x^5 + 4x^4 + 4x^3$

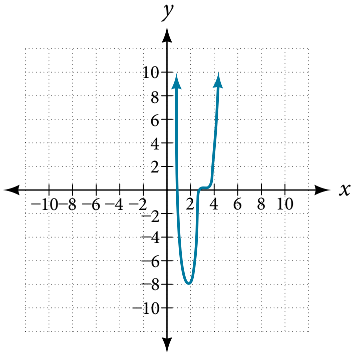
15. $f(x) = x^3 - 4x^2 + x - 4$

For the following exercises, based on the given graph, determine the zeros of the function and note multiplicity.

16.



17.



18. Use the Intermediate Value Theorem to show that at least one zero lies between 2 and 3 for the function $f(x) = x^3 - 5x + 1$

DIVIDING POLYNOMIALS

For the following exercises, use long division to find the quotient and remainder.

19. $\dfrac{x^3 - 2x^2 + 4x + 4}{x - 2}$

20. $\dfrac{3x^4 - 4x^2 + 4x + 8}{x + 1}$

For the following exercises, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.

21. $\dfrac{x^3 - 2x^2 + 5x - 1}{x + 3}$

22. $\dfrac{x^3 + 4x + 10}{x - 3}$

23. $\dfrac{2x^3 + 6x^2 - 11x - 12}{x + 4}$

24. $\dfrac{3x^4 + 3x^3 + 2x + 2}{x + 1}$

ZEROS OF POLYNOMIAL FUNCTIONS

For the following exercises, use the Rational Zero Theorem to help you solve the polynomial equation.

25. $2x^3 - 3x^2 - 18x - 8 = 0$

26. $3x^3 + 11x^2 + 8x - 4 = 0$

27. $2x^4 - 17x^3 + 46x^2 - 43x + 12 = 0$

28. $4x^4 + 8x^3 + 19x^2 + 32x + 12 = 0$

For the following exercises, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.

29. $x^3 - 3x^2 - 2x + 4 = 0$

30. $2x^4 - x^3 + 4x^2 - 5x + 1 = 0$

RATIONAL FUNCTIONS

For the following exercises, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph of the function.

31. $f(x) = \dfrac{x + 2}{x - 5}$

32. $f(x) = \dfrac{x^2 + 1}{x^2 - 4}$

33. $f(x) = \dfrac{3x^2 - 27}{x^2 + x - 2}$

34. $f(x) = \dfrac{x + 2}{x^2 - 9}$

For the following exercises, find the slant asymptote.

35. $f(x) = \dfrac{x^2 - 1}{x + 2}$

36. $f(x) = \dfrac{2x^3 - x^2 + 4}{x^2 + 1}$

INVERSES AND RADICAL FUNCTIONS

For the following exercises, find the inverse of the function with the domain given.

37. $f(x) = (x - 2)^2, x \geq 2$

38. $f(x) = (x + 4)^2 - 3, x \geq -4$

39. $f(x) = x^2 + 6x - 2, x \geq -3$

40. $f(x) = 2x^3 - 3$

41. $f(x) = \sqrt{4x + 5} - 3$

42. $f(x) = \dfrac{x - 3}{2x + 1}$

MODELING USING VARIATION

For the following exercises, find the unknown value.

43. y varies directly as the square of x. If when $x = 3$, $y = 36$, find y if $x = 4$.

44. y varies inversely as the square root of x. If when $x = 25$, $y = 2$, find y if $x = 4$.

45. y varies jointly as the cube of x and as z. If when $x = 1$ and $z = 2$, $y = 6$, find y if $x = 2$ and $z = 3$.

46. y varies jointly as x and the square of z and inversely as the cube of w. If when $x = 3$, $z = 4$, and $w = 2$, $y = 48$, find y if $x = 4$, $z = 5$, and $w = 3$.

For the following exercises, solve the application problem.

47. The weight of an object above the surface of the earth varies inversely with the distance from the center of the earth.If a person weighs 150 pounds when he is on the surface of the earth (3,960 miles from center), find the weight of the person if he is 20 miles above the surface.

48. The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure P. A cylinder contains oxygen at a temperature of 310 degrees K and a pressure of 18 atmospheres in a volume of 120 liters. Find the pressure if the volume is decreased to 100 liters and the temperature is increased to 320 degrees K.

CHAPTER 5 PRACTICE TEST

Give the degree and leading coefficient of the following polynomial function.

1. $f(x) = x^3(3 - 6x^2 - 2x^2)$

Determine the end behavior of the polynomial function.

2. $f(x) = 8x^3 - 3x^2 + 2x - 4$

3. $f(x) = -2x^2(4 - 3x - 5x^2)$

Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

4. $f(x) = x^2 + 2x - 8$

Given information about the graph of a quadratic function, find its equation.

5. Vertex (2, 0) and point on graph (4, 12).

Solve the following application problem.

6. A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If 1,200 feet of fencing is available, find the maximum area that can be enclosed.

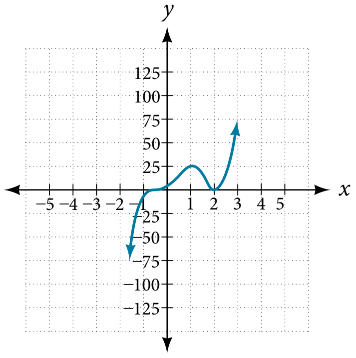
Find all zeros of the following polynomial functions, noting multiplicities.

7. $f(x) = (x - 3)^3(3x - 1)(x - 1)^2$

8. $f(x) = 2x^6 - 12x^5 + 18x^4$

Based on the graph, determine the zeros of the function and multiplicities.

9.



Use long division to find the quotient.

10. $\dfrac{2x^3 + 3x - 4}{x + 2}$

Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.

11. $\dfrac{x^4 + 3x^2 - 4}{x - 2}$

12. $\dfrac{2x^3 + 5x^2 - 7x - 12}{x + 3}$

Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.

13. $f(x) = 2x^3 + 5x^2 - 6x - 9$

14. $f(x) = 4x^4 + 8x^3 + 21x^2 + 17x + 4$

15. $f(x) = 4x^4 + 16x^3 + 13x^2 - 15x - 18$

16. $f(x) = x^5 + 6x^4 + 13x^3 + 14x^2 + 12x + 8$

Given the following information about a polynomial function, find the function.

17. It has a double zero at $x = 3$ and zeroes at $x = 1$ and $x = -2$. It's y-intercept is $(0, 12)$.

18. It has a zero of multiplicity 3 at $x = \frac{1}{2}$ and another zero at $x = -3$. It contains the point $(1, 8)$.

Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

19. $8x^3 - 21x^2 + 6 = 0$

For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.

20. $f(x) = \dfrac{x + 4}{x^2 - 2x - 3}$

21. $f(x) = \dfrac{x^2 + 2x - 3}{x^2 - 4}$

Find the slant asymptote of the rational function.

22. $f(x) = \dfrac{x^2 + 3x - 3}{x - 1}$

Find the inverse of the function.

23. $f(x) = \sqrt{x - 2} + 4$

24. $f(x) = 3x^3 - 4$

25. $f(x) = \dfrac{2x + 3}{3x - 1}$

Find the unknown value.

26. y varies inversely as the square of x and when $x = 3$, $y = 2$. Find y if $x = 1$.

27. y varies jointly with x and the cube root of z. If when $x = 2$ and $z = 27$, $y = 12$, find y if $x = 5$ and $z = 8$.

Solve the following application problem.

28. The distance a body falls varies directly as the square of the time it falls. If an object falls 64 feet in 2 seconds, how long will it take to fall 256 feet?